Topic list for the ADE qualifying exam

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Abstract

The applied differential equations qualifying exam ("ADE qual") is the responsibility of the applied research group, with frequent participation of members of the analysis research group. It corresponds to the graduate course sequence 266ABC, covering both ordinary and partial differential equations and emphasizing results from the first two courses in the sequence. Although these courses are typically taught in a self-contained way, which minimizes the expectation that students have been exposed to differential equations in upper-division undergraduate classes, it is not required that faculty teaching the 266 sequence review topics from undergraduate differential equations wholly or partly, and students should expect to review such background material as needed.

The following sections describe the prerequisites from undergraduate study and then the graduate-level topics for this exam. Students are encouraged to study these topics broadly, but they will have some ability to select problems on the exam. Specifically, the ADE qual has eight questions, which broadly cover ordinary differential equations, parabolic differential equations, hyperbolic differential equations, elliptic differential equations, and additional associated topics in nonlinear PDEs and variational calculus. Four essentially complete solutions (8/10 or above) will typically lead to a Pass. Scores that lie close to this threshold or that depend heavily on partial credit are assessed on a case-by-case basis.

1 Prerequisites

The ADE qual and corresponding courses assume working knowledge of analysis at about the undergraduate honors level, as well as undergraduate, upper-division level coverage of ordinary and partial differential equations. 266A and 266B instructors will often, but are not required to, review material from the second list.

- Concepts of continuity, including of sets or sequences of functions (i.e. uniform continuity, equicontinuity, and Lipschitz continuity).
- Concepts of convergence in sequences and series, including absolute and uniform convergence.
- Main classes of sets: open, compact, etc.
- Complex analysis for performing contour integrals (i.e. residue theorem, Jordan lemma, etc.), specifically in the context of integrals in Fourier and Laplace transform integrals and their inverses.
- Vector-space structure of solutions of linear ODEs.
- Diagonalizability of Hermitian matrices and existence of Jordan normal form.
- Introductory vector calculus: definition of gradient, curl, and divergence, and their computation in different coordinate systems. Integral theorems: Gauss's theorem, Green's formula in the plane, and Stokes' theorem.
- Separation of variables, integrating factors for ODEs.
- Fourier and Laplace transforms for solving ODEs.

- Phase portraits and vector-field plots for solving first-order autonomous ODES and systems of 2 first-order ODEs.
- Separation of variables for solving PDEs.
- Method of characteristics for solving first-order flux-conservation PDES.

This material is split over multiple textbooks. The analysis material is at the level of the basic qualifying exam, and students should consult the textbooks recommended there. For introductory differential-equations material and complex analysis, textbooks on mathematical methods in applied math (e.g. Riley, Hobson, Bence, or Boas) are recommended. For qualitative analysis of ODEs and systems of ODEs, Strogatz's book is recommended. For separation of variables to solve PDEs, see Haberman. See Section 4 for complete bibliographical information.

2 Ordinary differential equations and systems of ordinary differential equations

Ordinary differential equations are covered in 266A, but they also arise repeatedly in 266B, because many of the techniques for solving PDEs that are developed in that course (e.g. separation of variables and solution by characteristics) reduce the PDEs to systems of ordinary differential equations.

Topics:

- Analytical methods for solving ODEs. Equidimensional, scale-invariant, autonomous ODEs. Linear ODEs: Reduction of order; variation of parameters; undetermined coefficients; Wronskians. Inhomogeneous ODEs: Variation of parameters and Greens' function methods.
- Dominant balance and approximate solutions of ODEs.
- Theory and conditions for existence and uniqueness of solutions of initial-value problems: Picard–Lindelöf theorem. Cauchy–Peano.
- Solution by series expansions. Taylor series for ordinary points. Frobenius series for regular singular points. Carlini's transform and asymptotic analysis for essential singular points.
- Theory for self-adjoint boundary value problems: Construction of Green's functions and the Green's function as an inverse operator. Fourier's theorem and completeness of eigenfunctions.
- Sturm–Liouville theory. Case studies: Legendre polynomials, sine and cosine series, etc.
- Systems of linear ordinary differential equations. Solutions as exponentials.
- Nonlinear systems of ODEs, including both planar systems and systems with 3+ equations. Equilibria and their stability. Drawing phase portraits. Bifurcations.
- Conservative systems
- Reversible systems
- Lyapunov functions; weak and strong Lyapunov stability theorems. Instability theorem. La Salle's invariance principle.
- Limit cycles and existence theorems: Poincaré–Bendixson theorem.

3 Partial differential equations

Methods and theoretical results for partial differential equations are developed in 266BC. 266B begins study of PDEs with specific methods and theorems that are focused on key classes of equations: first-order equations (particularly Hamilton–Jacobi and transport equations) and the heat, wave, and Poisson equations. 266C introduces variational principles and the theory of weak solutions.

- Derivation of PDEs from conservation laws and force balances.
- Classification of linear second-order PDEs
- First-order PDEs and solution by method of characteristics. Quasilinear equations and nonlinear equations.
- Wave equation and related PDEs. D'Alembert's solution. Energy conservation. Uniqueness. Finite speed of propagation. Axially and spherically symmetric solutions. Solving wave equation in finite and semi-infinite domains.
- Heat equation. Energy. Green's function solutions. Properties of solutions: smoothness, backwards in-time solutions, etc. Solving in finite domains using separation of variables.
- Laplace's equation, Poisson's equation, and properties of harmonic functions. Variational formulation, including Dirichlet and Neumann boundary conditions. Smoothness, mean-value property, and strong maximum principle. Liouville's theorem. Solving in finite domains using separation of variables. Green's function solution and reformulation as integral equations.
- Eigenvalues and eigenfunctions of the Laplacian and related operators.
- Maximum principle for Laplace's equation, heat equation and similar.
- Duhamel's principle for inhomogeneous linear PDEs.
- Similarity solutions and their applications, including to the heat equation and porous-medium equation.
- 1D traveling-wave solutions.
- Theory of distributions and distributional derivatives, including for the Dirac delta function.
- Weak solutions and formalism of H^k spaces.
- Flux-conservation equations. Rarefaction waves. Integral and discontinuous solutions. Rankine–Hugoniot conditions. Non-uniqueness of integral solutions. Entropy conditions and their role in ensuring uniqueness. Viscosity solutions and specific application to Burgers' equation.
- Hamilton–Jacobi equations. Characteristic equations. Hopf–Lax formula.
- Euler-Lagrange equations and functional minimization. Lagrange multipliers. Variational interpretation of eigenvalues and eigenfunctions.

4 List of recommended readings and reference books

No single text corresponds exactly to the outline above, but many good texts cover these topics. The texts taht are used for 266ABC are chosen by the individual instructors. Below, a † denotes an introductory book that covers prerequisite material. Other books are taken from books that are commonly used to teach the 266ABC sequence.

Bender, Carl M., and Steven A. Orszag. *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory.* Springer-Verlag, 1999.

† Boas, Mary L. *Mathematical methods in the physical sciences.* John Wiley & Sons, 2006.

Coddington, Earl A., and Norman Levinson. *Theory of ordinary differential equations.* Tata McGraw-Hill Education, 1955.

Evans, Lawrence C. *Partial differential equations.* 2nd edition. Vol. 19. American Mathematical Soc., 2010.

† Haberman, Richard. *Applied partial differential equations with Fourier series and boundary value problems*, 5th Edition, Pearson, 2012.

Jordan, Dominic, and Peter Smith. *Nonlinear ordinary differential equations: an introduction for scientists and engineers.* OUP, Oxford, 2007.

† Riley, Kenneth Franklin, Michael Paul Hobson, and Stephen John Bence. *Mathematical methods for physics and engineering.* Cambridge University Press, 2006.

Shearer, Michael, and Rachel Levy. *Partial differential equations: An introduction to theory and applications.* Princeton University Press, 2015.

† Strogatz, Steven H. *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering.* CRC press, 2018.