1 The Basic: prerequisites from undergraduate study

The Basic Exam is usually attempted in early September of a PhD student's first year of study. But it is a challenging exam so our PhD program only requires that it be passed by the first quarter of the second year. It covers topics from undergraduate analysis and linear algebra which form a common core of mathematical knowledge, which all graduate students in our program are expected to master. This material is assumed in many of the graduate courses and the seven area qualifying exams.

The exam is four hours long, and has 12 questions, 6 in analysis and 6 in linear algebra. You are asked to answer 5 questions from each subject. Each question is worth 10 points. A score of at least 70 points on the exam, with at least 6 near complete (9 or 10 points) answers, including at least 3 in each of the areas (analysis, linear algebra), typically ensures a passing grade.

The students are encouraged to practice with the past qual exams: https://ww3.math.ucla.edu/past-qualifying-exams/

1.1 Basic Analysis

Topics

One-variable calculus

- Completeness of the real numbers
- Sequences, series, limits; familiarity with the difficulties around exchanging the order of limits; absolute convergence and rearrangements of series
- Continuous functions, including epsilon-delta arguments; maxima and minima; uniform continuity
- Definition of the derivative; the mean value theorem; Cauchy mean value theorem, L'Hopital's rule, uniform convergence limits of derivatives
- Rolle's theorem for higher order derivatives, including applications to error bounding for Lagrange interpolations.
- The Riemann integral including basic properties and integrability of sums, products, mins, maxes; mean value theorem for integrals; fundamental theorem of calculus; integration by parts; changes of variables; improper integrals; integrals of uniform convergence limits
- Sequences and series of functions; uniform convergence and integration; differentiation under the integral sign
- Taylor theorem with reminder in Lagrange, Cauchy, and integral forms
- Numerical integration with error estimation

Metric space topology and analysis (primarily in Euclidean spaces)

- Countability and uncountability (e.g. of the reals); fluency with examples of diagonal arguments

- Open and closed sets; interior and closure
- Completeness, including the Baire category theorem and some basic applications to subsets of the real line
- Sequences and convergence, continuity, in terms of the metric and the topology, uniform convergence and continuity
- Compactness, including sequential compactness, the Bolzano–Weierstrass theorem, the Heine-Borel theorem, totally bounded spaces.
- Connectedness and path connectedness
- Spaces of functions; modes of convergence for sequences of functions; the uniform norm on continuous functions as an important example of a metric space; uniform continuity; equicontinuity; Cauchy-Schwarz, Young, Hölder and Minkowski inequalities
- Formal power series, radius of convergence, absolute and uniform convergence on subintervals, derivatives and integrals of formal power series, real analytic functions, Abel's lemma and theorem, Fubini's theorem for sequences, multiplication of power series.
- Dini's theorem
- The Stone-Weierstrass theorem
- The Arzelà–Ascoli theorem
- The exponential and logarithm functions, sine and cosine, uniform approximation of periodic functions by trigonometric polynomials
- Contraction maps and fixed point theory, with applications to include at least (i) Newton's method and (ii) existence and uniqueness of solutions to non-linear ODEs

Multivariable calculus

- Definition of differentiability in several variables (approximating linear transformation); partial derivatives; directional derivatives; differentiability of functions with continuous partial derivatives; equality of mixed partials; local minima, maxima, and saddle points; chain rule; Taylor expansion in several variables and connection to Newton's method in several variables
- Inverse and implicit function theorems
- Lagrange multipliers
- Multivariable integration, including change of variables formula
- Differentiation under the integral sign
- Double integrals, line integrals, Green's theorem, Stokes theorem in $\mathbb{R}^3$, the divergence theorem in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Convex and concave functions: non-calculus definition and equivalent characterization using calculus; some examples

Suggested references

No single text corresponds exactly to the outline above, but many good texts cover these topics. Examples include:

- W. Rudin, *Principles of Mathematical Analysis*, McGraw–Hill
- C. Pugh, *Real Mathematical Analysis* (Chapters 1-5), Springer, available in hardcopy or online through Springerlink[1]:
  `https://link.springer.com/book/10.1007/978-0-387-21684-3`
- T. Gamelin and R. Greene, *Introduction to Topology*, Dover (mostly Chapter I)
- C.H. Edwards, *Advanced Calculus of Several Variables*, Dover (mostly Chapter I-III)
- T. Apostol, *Mathematical Analysis* (Chapters 1-5, 8, 9, 12-14), Addison–Wesley
- R. Rosenlicht, *Introduction to Analysis*, Dover

1.2 Basic Linear Algebra

Topics

Vector Spaces

- Fields (axiomatic definition and examples, esp. Q, R, C, finite fields).
- Vector spaces over a field.
- Subspaces and their complementary subspaces.
- Linear (in-)dependence, dimension, bases (including existence). Steinitz exchange theorem.
- Linear equations, Gauss elimination, reduced row echelon form (RREF).
- Dual spaces and dual bases. Quotient spaces.

Linear Transformations

- Matrix representations, change of basis, similarity of linear operators and square matrices.
- Trace of a square matrix, properties incl $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
- Image and kernel.
- Rank and nullity as well as rank-nullity theorem for linear transformations.

[1] Requires UCLA logon: if accessing off campus see
`https://www.library.ucla.edu/computers-computing-services/connect-campus`

- Induced maps on quotient spaces and dual maps between dual spaces.

Linear operators

- Eigenvalues and eigenvectors. Relationships between trace/determinant and eigenvalues.
- Diagonalizable operators. Examples of non-diagonalizable operators in all dimensions > 1.
- Generalized eigenspaces.
- Characteristic and minimal polynomials.
- Algebraic and geometric multiplicity of eigenvalues. Equality of algebraic multiplicity and the dimension of generalized eigenspaces.
- Cayley-Hamilton Theorem.
- Existence of eigenvalues/vectors, diagonalizablity over C iff minimal polynomial has no repeated roots.
- Rational and Jordan canonical forms. Relationship between the multiplicity of a root of the minimal polynomial and the corresponding largest Jordan block.

Inner product spaces

- Operator and Frobenius norms for operators.
- Orthogonal and orthonormal bases, Gram-Schmidt process.
- Adjoint transformations and (conjugate) transposes of matrices. Relationships between image/kernel of a transformation and the kernel/image of its adjoint.
- Orthogonal and unitary matrices and transformations.
- Self-adjoint operators (symmetric and hermitian) and their matrix representations.
- Diagonalizability of self-adjoint operators wrt orthonormal bases (spectral theorem).
- Normal operators and their canonical forms over R and C, canonical forms for orthogonal and unitary matrices.
- Quadratic forms: matrix representations, positive (semi-)definiteness, signature, Sylvester's criterion, and Sylvester's law of inertia.
- Matrix decompositions: QR, LU (including Choleski). Singular and Polar Decompositions.

Determinants.

- Basic theory and results incl Laplace expansion.
- Axiomatic characterization as multilinear function. Connection with signed volume.

- Computation via elementary operations or eigenvalues.
- Classical adjoints and Cramer's rule.

Linear Differential Equations

- Matrix exponentials and their use in solving first order linear ODEs.
- Higher order Linear ODEs.
- Wronskian for inhomogeneous higher order ODEs.

Suggested references

No single text corresponds exactly to the outline above, but many good texts cover these topics. Examples include:

- Peter Petersen, *Linear Algebra*, Springer, available in hardcopy or online through Springerlink[2]: `https://link.springer.com/book/10.1007%2F978-1-4614-3612-6`
- Serge Lang, *Linear Algebra*, Springer, available in hardcopy or online through Springerlink[2]: `https://link.springer.com/book/10.1007%2F978-1-4757-1949-9`
- Sheldon Axler, *Linear Algebra Done Right*, Springer, available in hardcopy or online through Springerlink[2]: `https://link.springer.com/book/10.1007/978-3-319-11080-6`
- Gilbert Strang, *Linear Algebra and its Applications*, Academic Press (a bit elementary but with applications)
- Denis Serre, *Matrices: Theory and Applications*, Springer, available in hardcopy or online through Springerlink[2]: `https://link.springer.com/book/10.1007%2F978-1-4419-7683-3`
- Kenneth Hoffmann and Ray Kunze, *Linear Algebra*, Pearson
- Horn and Johnson, *Matrix Analysis*, Cambridge University Press.

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