Logic Qualifying Exam

The Logic qualifying exam consists of eight questions, on computability theory, model theory, set theory, and incompleteness. Each question is worth 10 points. A score of at least 50 points, with complete solutions to at least one question in each of computability theory, model theory, and set theory, is usually enough for a pass.

Outline of material in model theory:

1. First-order logic: syntax and semantics, formal proofs, Gödels completeness theorem.
2. Compactness, Löwenheim-Skolem, the Diagram Method, Quantifier Elimination, Back-and-Forth method.
3. Completeness and quantifier elimination for some important theories: Algebraically Closed / Real Closed Fields, Presburger Arithmetic.
4. Ultrafilters and Ultraproducts, a proof of the Compactness Theorem using ultraproducts.
5. Spaces of types, Omitting Types Theorem.
6. Special types of models: prime and atomic, homogeneous, saturated; definability and the automorphism groups of saturated models.
7. The number of countable models, countably categorical theories (Ryll-Nardziewski theorem) and Vaught's "no two models" theorem.
8. Uncountably Categorical Theories, omega stable theories, strongly minimal theories, ideas of the proof of Morley's categoricity theorem and the Baldwin-Lachlan Theorem.

References:

- Hils, Martin; Loeser, Franois, *A first journey through logic.* Student Mathematical Library, 89. American Mathematical Society, Providence, RI, 2019.

  Chapters 2, 3.

- Marker, David, *Model theory. An introduction.* Graduate Texts in Mathematics, 217. Springer-Verlag, New York, 2002.

  Chapters 2, 3, 4.1–4.4, 6.1.

- Chang, C. C.; Keisler, H. J., *Model theory.* Third edition. Studies in Logic and the Foundations of Mathematics, 73. North-Holland Publishing Co., Amsterdam, 1990.

  Chapter 4.1.

- Chernikov, Artem, *Lectures on model theory, 220A Fall 2021.* Available at `https://youtube.com/playlist?list=PL54Pt_mZzBqibWHgesgEICeQHnwHom8xz`

Outline of material in computability theory and incompleteness:

1. Definition of computability, universal machines, indices for partial computable functions and c.e. sets, the s-m-n theorem, the recursion theorem.
2. The incomputability of the halting problem, Rice's theorem.
3. Notions of reducibilities, Myhill's isomorphism theorem.
4. Incomputability of truth in the natural numbers, representability in PA, Gödel-Rosser incompleteness, Gödel's second incompleteness theorem, Löb's theorem, Tennenbaum's theorem.
5. Relative computability, Turing reducibility, the Turing jump, the Kleene-Post Theorem, minimal pairs of Turing degrees, 1-generic reals, Friedberg jump inversion, exact pairs.
6. Finite injury priority constructions, Post's problem, the Friedberg-Muchnik theorem.
7. Post's hierarchy theorem, the arithmetical hierarchy, the limit lemma, completeness in the arithmetical hierarchy, relativization.
8. The Borel hierarchy, universal sets in the Borel hierarchy, Wadge reducibility, analytic and coanalytic sets, the Souslin-Kleene theorem.

References:

- Soare, Robert I., *Turing computability. Theory and applications. Theory and Applications of Computability.* Springer-Verlag, Berlin, 2016.

  Chapters 1, 2.1–2.3, 3.1–3.6, 4.1–4.3, 6.1–6.5, 7.1–7.3.

- Kaye, Richard, *Models of Peano arithmetic.* Oxford Logic Guides, 15. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.

  Chapters 1, 2, 3, 5.1–2, 6, 11.

- Kechris, Alexander S., *Classical descriptive set theory.* Graduate Texts in Mathematics, 156. Springer-Verlag, New York, 1995.

  Chapters 11, 14, 22.

Outline of material in set theory

1. The Zermelo-Fraenkel axioms and Axiom of Choice, basic consequences of ZF including ordinals, induction, recursion, trasfinite induction, transfinite recursion, wellorderings, rank functions, ordertypes, cardinals, basic cardinal arithmetic, Schroeder-Bernstein and Cantor's theorems, every cardinal has a successor, equivalence of AC and the wellordering principle.

2. Simple combinatorics on countable ordinals, regular and singular cardinals, clubs and stationary sets, pressing down lemma.

3. Basic framework for equiconsistency proofs, formula relativization, absoluteness of Δ_0 formulas in transitive models, the von Neumann hierarchy, equiconsistency of $\mathsf{ZF} - \mathsf{Foundation}$ and ZF.

4. Gödel's constructible universe L, the Reflection Theorem, ZF in L, L can be wellordered, absoluteness of provably Δ_1 notions, absoluteness of notions defined by recursion and transfinite recursion, AC in L, equiconsistency of ZF and ZFC and $\mathsf{ZFC}+V = L$, the model $L(A)$ of sets constructible starting from a transitive set A.

5. Mostowski collapse, condensation in L, the GCH in L.

6. The combinatorial principle $\diamond$ and its variants, $\diamond$ in L, Aronszajn trees, Suslin trees in L, Suslin lines and failure of Suslin Hypothesis in L.

7. Measurable cardinals, ultrapowers and ultrapower embeddings, no measurables in L.

8. Determinacy of closed games, Π^1_1 determinacy from a measurable, a Δ^1_2 wellordering of the reals in L, failure of Π^1_1 determinacy in L.

References:

- Kenneth Kunen, *Set Theory, an introduction to independence proofs.* Studies in logic and the foundations of mathematics, volume 102. North-Holland Publishing Company, 1980.

  Chapters 1, 2.4–2.7, 3, 4, 5.1, 6.

- Thomas Jech, *Set Theory, the third millennium edition, revised and expanded.* Springer Monographs in Mathematics. Springer, 2002.

  Chapter 10 sections on Measurable and real-valued measurable cardinals, Measurable cardinals, Normal measures, Chapter 17 section on Ultrapowers and elementary embeddings, Chapter 25 section on Projective sets and constructibility, Chapter 33 section on Projective determinacy.