Some Variational Problems Arising in Image Processing
2 The Functional Properties

We now turn to the question of determining analytical expressions for the functional properties of the system. We assume that the system is a linear, time-invariant system, and we consider the following transfer function,

\[
H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

where \(\zeta\) is the damping ratio and \(\omega_n\) is the natural frequency.

To find the impulse response, we compute the inverse Laplace transform of \(H(s)\),

\[
\mathcal{L}^{-1}\{H(s)\} = h(t) = 2\zeta \omega_n e^{-\zeta \omega_n t} \sin(\sqrt{1 - \zeta^2} \omega_n t)
\]

The frequency response is obtained by evaluating \(H(j\omega)\) for all frequencies \(\omega\),

\[
H(j\omega) = \frac{1}{\sqrt{\omega_n^2 - \omega^2}} e^{-\zeta \sqrt{\omega_n^2 - \omega^2} \omega}
\]

The phase response is the phase angle of \(H(j\omega)\),

\[
\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_n}\right)
\]

Finally, the group delay is given by the negative derivative of the phase response with respect to frequency,

\[
\text{Group Delay} = \frac{d\angle H(j\omega)}{d\omega} = \frac{\omega_n^2}{\omega_n^2 - \omega^2}
\]
\[ \int (x^n + x^m + x^p) \frac{d}{dx} \left( x^n + x^m + x^p \right) \]
\textbf{Theorem 2.2 (Rational Division)}

\begin{equation}
\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} a_n \left( \frac{x}{b} \right)^n
\end{equation}

where \( a_n = \frac{f^{(n)}(0)}{g^{(n)}(0)} \) for \( n \geq 0 \).

\textbf{Proof:}

Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) be the Taylor series of \( f \) and \( g \) around \( z = 0 \). Then, the quotient \( \frac{f(z)}{g(z)} \) is given by the power series

\begin{equation}
\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} c_n z^n
\end{equation}

where

\begin{equation}
c_n = \frac{1}{n!} \sum_{k=0}^{n} \frac{a_k}{b_{n-k}}
\end{equation}

for \( n \geq 0 \).

\section{Applications}

The theorem has applications in various fields such as numerical analysis and signal processing. For example, in numerical analysis, rational functions are used to approximate solutions to differential equations. In signal processing, they are used to design filters for signal transmission.

\begin{example}
Consider the function \( f(x) = \frac{1}{1 + x^2} \). Its Taylor series expansion around \( x = 0 \) is

\begin{equation}
\sum_{n=0}^{\infty} (-1)^n x^{2n}
\end{equation}

which converges for \( |x| < 1 \).
\end{example}
\[ \frac{(a-f) + b}{(a-f) + b} = \frac{c}{d} \]

(22)

To analyze the properties of the given expression, let:

\[ a = \{n \in \mathbb{N} \mid (n = n) \} \]

Proof:

The above equation holds true under certain conditions. Let's analyze the expression further:

\[ \left| \frac{a}{b} \right| = \left| \frac{c}{d} \right| \]

(23)

The expression is valid under the following conditions:

- \( a, b, c, d \) are integers.
- \( b \neq 0 \) and \( d \neq 0 \).

We conclude:

\[ \left( \frac{a}{b} \right) + \left( \frac{c}{d} \right) = \left( \frac{e}{f} \right) \]

Condition:

\[ \left| \frac{a}{b} \right| \leq \frac{1}{2} \]

(24)

Example:

Consider the following scenario:

\[ \left( \frac{a}{b} \right) + \left( \frac{c}{d} \right) = \left( \frac{e}{f} \right) \]

where:

- \( a, b, c, d, e, f \) are non-zero integers.
- \( b \neq 0 \) and \( d \neq 0 \).

We find:

\[ \left( \frac{a}{b} \right) + \left( \frac{c}{d} \right) = \left( \frac{e}{f} \right) \]

\[ \left| \frac{a}{b} \right| \leq \frac{1}{2} \]

Condition:

\[ \left| \frac{a}{b} \right| \leq \frac{1}{2} \]

(25)

The above expression holds true under the specified conditions.

\[ \left( \frac{a}{b} \right) + \left( \frac{c}{d} \right) = \left( \frac{e}{f} \right) \]

where:

- \( a, b, c, d, e, f \) are non-zero integers.
- \( b \neq 0 \) and \( d \neq 0 \).

We conclude:

\[ \left| \frac{a}{b} \right| \leq \frac{1}{2} \]

Condition:

\[ \left| \frac{a}{b} \right| \leq \frac{1}{2} \]

(26)

The above expression holds true under the specified conditions.
Theorem 1. Type I is Type II and vice versa.

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} f(k)}{n} = \int f(x) \, dx
\]

and

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} g(k)}{n} = \int g(x) \, dx
\]

The function \( f(x) \) is Type I and \( g(x) \) is Type II, and vice versa.

Theorem 2. Under certain conditions, Type I and Type II functions are equivalent.

\[
\frac{\sum_{k=1}^{n} f(k)}{n} \text{ is Type I } \Leftrightarrow \frac{\sum_{k=1}^{n} g(k)}{n} \text{ is Type II}
\]

\[
\int f(x) \, dx \text{ is Type I } \Leftrightarrow \int g(x) \, dx \text{ is Type II}
\]

Theorem 3. For a given function, the Type I and Type II representations are equivalent.

\[
\frac{\sum_{k=1}^{n} f(k)}{n} = \int f(x) \, dx
\]

\[
\int g(x) \, dx = \int \frac{\sum_{k=1}^{n} g(k)}{n} \, dx
\]

By the equivalence of Types I and II, the following conclusions hold:

\[
\frac{\sum_{k=1}^{n} f(k)}{n} = \int f(x) \, dx = \int \frac{\sum_{k=1}^{n} g(k)}{n} \, dx
\]

\[
\text{Type I} \Leftrightarrow \text{Type II}
\]

\[
\int f(x) \, dx = \int g(x) \, dx
\]
Exercise of minima

A condition for the existence of a minimum is

\[ \delta \int_c^d f(x) \, dx = \sum_j \left( b_j - a_j \right) f(x_j) \]

for a partition of \( [a, b] \) into \( n \) subintervals.

2. Boundary extremals

Consider a function \( f(x) \) defined on \( [a, b] \) such that

\[ \int_a^b f(x) \, dx \]

is an extremum.

For a comparison on \( [a, b] \), we have

\[ \frac{d}{dx} \left( \int_a^x f(t) \, dt \right) = f(x) \]

for any interval \( (a, b) \).
In the particular case of finite difference approximation of convolution kernels,

\[ f \ast k = (f \ast k) \text{ for } k \in \mathbb{R}^d \text{ and } f \in \mathbb{R}^d. \]

The convolution operator \( \ast \) and partition case

\[ l = d = d \text{ for } \in \mathbb{R}^d \text{ and } g \in \mathbb{R}^d. \]

Theorem 2.2

\[ l = d = d \text{ for } \in \mathbb{R}^d \text{ and } g \in \mathbb{R}^d. \]

The convolution operator \( \ast \) and partition case

\[ l = d = d \text{ for } \in \mathbb{R}^d \text{ and } g \in \mathbb{R}^d. \]
4. Critique: A possible error in the statement is that it does not specify the conditions under which the inequality holds. It is essential to ensure that the function and the parameters are such that the inequality is meaningful. For instance, if the function $f$ is not defined for all $x$, then the inequality may not hold for certain values of $x$. Therefore, a more appropriate statement might be that the inequality holds under certain conditions that are not explicitly stated in the given text.
Figure 7. A decomposition of $f$ from Figure 4 using the model (37) with the kernel being the Gaussian kernel of standard deviation $\sigma = 1$ and $\lambda = 1$.

Figure 6. A decomposition of $f$ from Figure 4 using the model (37) with the kernel being the characteristic function of a region centered at 0 having radius $1$, for the sides, and $\lambda = 1.5$.

Figure 9. A decomposition of $f$ from Figure 4 using the model (38) with the kernel being the Gaussian kernel of standard deviation $\sigma = 0.5$, $A = 1$.

Figure 8. A decomposition of $f$ from Figure 4 using the model (38). Here the medially component.