ADE Exam, Spring 2021
Department of Mathematics, UCLA

1. Find and classify all of the equilibrium points of

$$\ddot{\theta} + b\dot{\theta} + \sin(\theta) = 0 \tag{1}$$

for all $b > 0$, and plot the phase portraits for the qualitatively different cases.

2. Find the complete asymptotic series of the solutions to the equation

$$t^2\ddot{y} + t\dot{y} - (t^2 + \nu^2)y = 0 \tag{2}$$

as $t \to \infty$. For what values of ν does the series terminate after finitely many terms?

3. The height $h(r,t) \geq 0$ of a spreading axisymmetric droplet obeys the following equation:

$$\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right) \quad , \quad 0 < r < L(t),$$

along with the boundary condition $h(L(t),t) = 0$ and volume constraint $\int_0^{L(t)} rh\,dr = 1$. Here r is the distance from the center of the drop and $\{r \leq L(t)\}$ equals the support of the drop at time t.

Assume that h is a similarity solution of the form $h(r,t) = t^\alpha H\left(r/t^\beta\right)$ and $L(t) = \eta_0 t^\beta$, and solve the differential equation to find the function H and the constants α and β explicitly. We assume that h is smooth in its support.

You should present an equation solved by η_0, though it is not necessary to solve for η_0 explicitly.

4. A field $u : [0,\infty)^2 \times [0,\infty) \to \mathbb{R}$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

with boundary conditions: $u = 0$ on $x_1 = 0$ and $\frac{\partial u}{\partial x_1} = 0$ on $x_2 = 0$.

Suppose that at $t = 0$, the functions $u(\cdot,0)$ and $\frac{\partial u}{\partial t}(\cdot,0)$ are compactly supported on $\sqrt{x_1^2 + x_2^2} < 1$. Find the compact support for u at time t.

If you make use of results for the domain of dependence of a solution of the wave equation, then you should prove them.

5. Consider the conservation law

$$\begin{cases} u_t + (u^3)_x = 0 & \text{in} \quad \mathbb{R} \times (0,\infty); \\ u = g & \text{on} \quad \mathbb{R} \times \{t = 0\}. \end{cases}$$

We refer to *entropy solutions* as integral solutions which satisfy the Lax entropy condition.

(a) Find the entropy solution when

$$g(x) = 1 \text{ for } 0 < x < 1, \text{ otherwise } g = 0.$$

(b) Find one integral solution of the above problem that is not an entropy solution.

6. Let $\Omega = \{|x| < 1\} \subset \mathbb{R}^n$ and for given continuous functions $g(x)$ and $u_0(x)$, let u be a smooth solution of

$$\begin{cases} u_t - \Delta(u^2) = 0 & \text{in} \quad \Omega \times (0,\infty); \\ u = g & \text{on} \quad \partial\Omega \times (0,\infty); \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases}$$

Let $0 < g, u_0 < 1$. Show that $0 < u < 1$.

7. For a bounded domain Ω in $\mathbb{R}^n$ and for

$$u \in \mathcal{A} := \{u \in C^1(\Omega) \text{ with } u = 0 \text{ on } \partial\Omega \text{ and } \int_\Omega u = 1\},$$

consider the energy

$$E(u) := \int_\Omega [\sqrt{1 + |Du(x)|^2} + |x|^2 u(x)]dx,$$

where $Du(x)$ denotes the spatial gradient of u.

(a) Show that $E(u)$ has at most one minimizer among $u \in \mathcal{A}$.

(b) Let $\Omega = \{|x| < 1\}$, show that the minimizer u^* in $\mathcal{A}$, if it exists, is radial, i.e. it depends only on the value of $|x|$.

8. The function $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ satisfies the equation:

$$u_t + \frac{1}{2}u_x^2 = 0 \ ,$$

with initial condition: $u(x,0) = \frac{1}{2(x^2+1)}$.

(a) Show that solution found by method of characteristics is only valid for a time interval $0 \le t < T$. What is the value of T?

(b) Derive a formula for $u(0,t)$ for all $t > 0$.