Applied Differential Equations

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper.
Write your university identification number at the top of each sheet of paper.

DO NOT WRITE YOUR NAME!
Complete this sheet and staple to your answers. Read the directions of the exam very carefully.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE
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1. ______________________   5. __________________________

2. ______________________   6. __________________________

3. ______________________   7. __________________________

4. ______________________   8. __________________________

Pass/fail recommend on this form.

Total score: ____________________

Form revised 3/08

ADE Exam, Spring 2022
Department of Mathematics, UCLA

1. [10 points] Prove that the origin is a center (in the full nonlinear system) for the dynamical system

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + f(x)\frac{\mathrm{d}x}{\mathrm{d}t} + g(x) = 0 \tag{1}$$

if the following conditions hold:

(a) $f(x)$ is odd and $f(x) > 0$ when $x > 0$;

(b) $g(x) > 0$ for $x > 0$, and $g(x)$ is odd;

(c) $g(x) > \alpha f(x)F(x)$ for $x > 0$, where $F(x) = \int_0^x f(u)\,\mathrm{d}u$ and $\alpha > 1$.

In proving this result, note that $g'(0) = 0$ may hold.

[To help with intuition on this problem, note that damping is positive (leading to energy loss) when $f(x) > 0$ and that damping is negative (leading to energy gain) when $f(x) < 0$.]

2. [10 points] Consider the ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x)\,. \tag{2}$$

(a) By considering (2) with an appropriate initial condition and a right-hand side of

$$f(t,x) = \begin{cases} 0\,, & t = 0\,,\ -\infty < x < \infty \\ 2t\,, & 0 < t \leq 1\,,\ -\infty < x < 0 \\ 2t - \frac{4x}{t}\,, & 0 < t \leq 1\,,\ 0 \leq x \leq t^2 \\ -2t\,, & 0 < t \leq 1\,,\ t^2 < x < \infty\,, \end{cases} \tag{3}$$

demonstrate explicitly that continuity of the right-hand side $f(t, x)$ alone is not sufficient to guarantee convergence of successive Picard approximations to a solution of (2).

(b) If the right-hand side of (2) is continuous and we know that there is a unique solution of (2) on an interval, is that sufficient in general to guarantee convergence of successive Picard approximations to that solution?

(c) If successive Picard approximations to (2) converge to a solution, is it necessarily true that that solution is unique?

3. [10 points] Suppose that $\mathbf{A}$ is a symmetric, positive definite $n \times n$ matrix and that $\mathbf{c}$ is a smooth vector field defined on a bounded open set $U \subseteq \mathbb{R}^n$, with piecewise smooth and orientable boundary ∂U. Show that the partial differential equation

$$-\sum_{i,j} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i c_i \frac{\partial u}{\partial x_i} = 0\,,$$

with boundary condition $u(x) = g(x)$ for $x \in \partial U$, has at most one solution that is $\mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$.

4. [10 points] Let $u(x,t;y)$, with $x,y,t>0$, be a Green's function solution of the partial differential equation

$$\frac{\partial u}{\partial t} = u + \frac{\partial^2 u}{\partial x^2}\,,$$

with boundary conditions $u(0,t;y)=0$, $u(\infty,t;y)=0$ and initial condition $u(x,0;y)=\delta(x-y)$. By explicitly deriving a formula for the solution u, show that it satisfies the reciprocity property $u(x,t;y)=u(y,t;x)$.

[Note: If you make use of the fundamental solution of the heat equation in your solution, then you should state its formula. However, you do not need to prove it.]

5. [10 points] Suppose that $u(x,t)$ satisfies the porous-medium partial differential equation on an expanding domain:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^2 \frac{\partial u}{\partial x}\right), \quad |x| < L(t)\,,\ t>0\,, \tag{4}$$

with boundary conditions $u(\pm L(t),t)=0$ and total-mass constraint $\int_{-L(t)}^{L(t)} u \,\mathrm{d}x = 1$.

Find a similarity solution of (4) of the form $u(x,t)=t^a f\left(\frac{x}{t^b}\right)$ and $L(t)=ct^b$. Your solution should include the values of a, b, and c and an explicit expression for the function f.

6. [10 points] Using the method of characteristics, solve

$$xu_x + yu_y = u\,, \quad u(x,1) = \frac{1}{1+x^2}$$

for $u(x,y)$, with $y>0$.

7. [10 points] Let $u \in C^2(R \times [0,\infty))$.

(a) Solve the initial-value problem for the wave equation in one dimension:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \quad \text{in} \quad \mathrm{R}\times(0,\infty)\,, \\ u = g\,, \quad u_t &= h \quad \text{in} \quad \mathrm{R}\times\{\mathrm{t}=0\}\,. \end{aligned} \tag{5}$$

(b) Suppose that g and h have compact support. The kinetic energy is $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,\mathrm{d}x$ and the potential energy is $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,\mathrm{d}x$.

Prove the following statements:

(i) $k(t)+p(t)$ is a constant, where you should write the constant in terms of the initial data;

(ii) $k(t)-p(t)$ is time independent for sufficiently large t.

8. [10 points] Consider the following hyperbolic conservation law for traffic flow:

$$u_t + (u(1-u))_x = 0\,, \tag{6}$$

where u is the density of vehicles and $1-u$ is the mean speed of vehicles at density u. Note that the flux of vehicles is the mean speed multiplied by the mean density.

(a) Solve the Riemann problem for the case of vehicles stopped at a traffic light that turns green at time $t = 0$:

$$u(x,0) = \begin{cases} 1\,, & x < 0 \\ 0\,, & x \geq 0\,. \end{cases} \tag{7}$$

(b) Solve the Riemann problem for the case of congestion on a road:

$$u(x,0) = \begin{cases} 0.25\,, & x < 0 \\ 0.75\,, & x \geq 0\,. \end{cases} \tag{8}$$

(c) In the congestion case in part (b), suppose that you are traveling in a vehicle whose speed is the mean speed that is described above. Your vehicle's position starts at $x = -1$ at time $t = 0$.

What is your path $x(t)$ going forward in time?

[Hint: This is not a characteristic. Use the solution to (b) to determine the velocity of your vehicle before and after it enters the congestion region.]