Algebra Qualifying Exam

Read the instructions of the exam carefully.

Complete this sheet and staple to your answers.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW

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Total score: ____________________

Pass/fail recommend on this form.

Revised 3/30/2010

ALGEBRA QUALIFYING EXAM

2022 MARCH

All answers must be justified. State clearly any theorem that you use.

Problem 1. Let F be a field of characteristic not 2 and let the symmetric group S_n act on the polynomial ring $F[X_1, \dots, X_n]$ by permuting the variables, for $n \geq 2$. Let $A = (F[X_1, \dots, X_n])^{A_n}$ and $B = (F[X_1, \dots, X_n])^{S_n}$ be the fixed subrings, where $A_n < S_n$ is the alternating group.

(a) Show that A is an integral extension of B.
(b) Show that $A = B[\delta]$ for some $\delta \in A$ such that $\Delta := \delta^2$ belongs to B.
(c) For $n = 2$, describe Δ as a polynomial in $e_1 = X_1 + X_2$ and $e_2 = X_1X_2$.

Problem 2. Let R be a ring, $S_1 = (0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0)$ a short exact sequence of right R-modules and $S_2 = (0 \to L \xrightarrow{h} M \xrightarrow{k} N \to 0)$ a short exact sequence of left R-modules in which M is free. Show that if

$$Z \otimes_R S_2 = (0 \to Z \otimes_R L \to Z \otimes_R M \to Z \otimes_R N \to 0)$$

is exact then the sequence $S_1 \otimes_R N$ is exact as well.

Problem 3. Let G be a finite p-group and let $H < G$ be a proper subgroup. We write as usual $H^g = g^{-1}Hg$ for every $g \in G$.

(a) Show that the normalizer $N_G(H)$ of H in G is strictly larger than H.
(b) Show that if H is not normal in G then there exists another proper subgroup $H < K < G$ and $g \in G$ such that $K^g = K$ but $H^g \neq H$.

Problem 4. Let R be a commutative ring and M be an R-module.

(a) Show that $\mathrm{Hom}_R(-, M)\colon (R\text{-}\mathsf{Mod})^{\mathrm{op}} \to R\text{-}\mathsf{Mod}$ admits a left adjoint.
(b) Show that for every R-module X, the module $\mathrm{Hom}_R(X, M)$ is a direct summand of $\mathrm{Hom}_R\big(\mathrm{Hom}_R\big(\mathrm{Hom}_R(X, M), M\big), M\big)$.

Problem 5. Let k be a commutative ring and let G be a finite group. Prove that k with trivial G action is a projective kG-module if and only if the order of G is invertible in k. (If you learned this as a theorem, give its proof.)

Problem 6. Let G be a group of order 30.

(a) Prove that G contains an element of order 15.
(b) Prove that G is the semidirect product of cyclic subgroups of order 15 and 2.

Problem 7. Let K/F be a finite separable field extension, and let L/F be any field extension. Show that $K \otimes_F L$ is a product of fields.

Problem 8. A nonzero idempotent $e = e^2$ in a commutative ring is called primitive if it cannot be written as the sum of two nonzero idempotents x and y such that $xy = 0$. Prove that every nonzero Noetherian commutative ring admits a primitive idempotent.

Problem 9. Let A be a (unital) algebra of dimension n over a field F. Prove that there is a (unital) F-algebra homomorphism from $A \otimes_F A^{\text{op}}$ to the F-algebra of $n \times n$ matrices, where A^{op} is the opposite algebra.

Problem 10. Let F be a field characteristic not 2 and let $K = F(\sqrt{a}, \sqrt{b})$ be a biquadratic field extension (of degree 4) of F, for $a, b \in F^\times$ not squares. Suppose that $b = x^2 - ay^2$ for some $x, y \in F$ (i.e., b is a norm for the quadratic extension $F(\sqrt{a})/F$). Prove that there is a field extension L of K that is Galois over F with Galois group the dihedral group of order 8.