Analysis

Read the instructions of the exam carefully.

Complete this sheet and staple to your answers.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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Pass/Fail recommendation on this form.

Total score: ___________________

Form revised 3/08

ANALYSIS QUAL: MARCH 25, 2021

Please be reminded that to pass the exam you need to show mastery of both real and complex analysis. Please choose at most 10 questions to answer, including at least 4 from problems 1–6 and 4 from problems 7–12. On the front of your paper indicate which 10 problems you wish to have graded.

Problem 1. Let μ be a positive Borel probability measure on $[0,1]$ and let

$$C = \sup\Big\{\mu(E) : E \subset [0,1] \text{ with } |E| = \tfrac{1}{2}\Big\}$$

where $|E|$ denotes the Lebesgue measure of E. Show that there exists a Borel set $F \subset [0,1]$ such that

$$|F| = \tfrac{1}{2} \quad \text{and} \quad \mu(F) = C.$$

Hint. When $d\mu = f\,dx$, one can sometimes take $F = \{x \in [0,1] : f(x) > \lambda\}$, for a suitable $\lambda \geq 0$.

Problem 2. Let μ and ν be two finite positive Borel measures on $\mathbb{R}^n$.

a) Suppose that there exist Borel sets $A_n \subset X$ so that

$$\lim_{n\to\infty} \mu(A_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \nu(X \setminus A_n) = 0.$$

Show that μ and ν are mutually singular.

b) Suppose there are non-negative Borel functions $\{f_n\}_{n\geq 1}$ so that $f_n(x) > 0$ for ν-a.e. x and

$$\lim_{n\to\infty} \int f_n(x)\,d\mu(x) = 0 \quad \text{and} \quad \lim_{n\to\infty} \int \frac{1}{f_n(x)}\,d\nu(x) = 0.$$

Show that μ and ν are mutually singular.

Problem 3. Let $f \in L^2(\mathbb{R})$. For $n \geq 1$ we define

$$f_n(x) = \int_0^{2\pi} f(x+t)\,\cos(nt)\,dt.$$

Prove that f_n converges to zero *both* almost everywhere in $\mathbb{R}$ *and* in the $L^2(\mathbb{R})$ topology, as $n \to \infty$.

Problem 4. Define

$$I(f) := \int_0^1 \left(\frac{1}{2}\left(f'(x)\right)^2 + \sin\left(f(x)\right) + f^4(x)\right) dx$$

for any $f \in C^1([0,1];\mathbb{R})$. Let $f_n \in C^1([0,1];\mathbb{R})$ be such that

$$I(f_n) \to \inf_{f \in C^1([0,1];\mathbb{R})} I(f).$$

Show that the sequence $\{f_n\}$ has a limit point in the space $C([0,1];\mathbb{R})$.

Problem 5. Let $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ be such that the series

$$\sum_{i=1}^{\infty} x_i y_i$$

converges for all $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$ such that $\lim_n y_n = 0$. Show that the series $\sum_{n=1}^{\infty} |x_n|$ converges.

Problem 6. We say that a linear operator $T : C([0,1]) \to C([0,1])$ is *positive* if $T(f)(x) \geq 0$ for all $x \in [0,1]$, whenever $f \in C([0,1])$ satisfies $f(x) \geq 0$ for all $x \in [0,1]$. Let

$$T_n : C([0,1]) \to C([0,1])$$

be a sequence of positive linear operators such that $T_n(f) \to f$ uniformly on [0,1] if f is a polynomial of degree less than or equal to 2. Show that

$$T_n(f) \to f \quad \text{uniformly on } [0,1],$$

for every $f \in C([0,1])$.

Hint. Let $f \in C([0,1])$. Show first that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(x) - f(y)| \leq \varepsilon + C_\varepsilon |x-y|^2 \quad \text{for all } x, y \in [0,1].$$

Problem 7. Let $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}$. Show that there exists a *unique* bounded harmonic function $u : \Omega \to \mathbb{R}$ such that for all $x > 0$ and $y > 0$,

$$\lim_{t \to 0} u(x + it) = 0 \quad \text{and} \quad \lim_{t \to 0} u(t + iy) = 1.$$

Problem 8. Show that there exists a non-zero entire function $f : \mathbb{C} \to \mathbb{C}$ and constants $b, c \in \mathbb{C}$ satisfying

$$f(0) = 0, \quad f(z+1) = e^{bz} f(z), \quad \text{and} \quad f(z+i) = e^{cz} f(z).$$

Problem 9. Let $\Omega_1 \subseteq \Omega_2$ be bounded Jordan domains in $\mathbb{C}$. We also assume that $0 \in \Omega_1$. Now suppose $f_1 : \mathbb{D} \to \Omega_1$ and $f_2 : \mathbb{D} \to \Omega_2$ are Riemann mappings, satisfying $f_1(0) = f_2(0) = 0$. Show that

$$|f_1'(0)| \leq |f_2'(0)|$$

Problem 10. Define

$$f(z) = \int_0^1 \frac{t^z}{e^t - 1} dt, \quad z \in \mathbb{C}, \ \operatorname{Re} z > 0.$$

Show that f is an analytic function in $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and that it admits a meromorphic continuation $\widehat{f}$ to the region $\{z \in \mathbb{C} : \operatorname{Re} z > -1\}$. Compute the residue of $\widehat{f}$ at $z = 0$.

Problem 11. For an entire function $f(z) = f^{(0)}(z)$, we define

$$f^{(n)}(z) = f(f^{(n-1)}(z)) \quad \text{for all} \quad n \geq 1$$

a) Show that if there exists an $n \geq 1$, such that $f^{(n)}$ is a polynomial, then f is a polynomial.

b) Prove that for any $n \geq 1$ we have $f^{(n)}(z) \neq e^z$.

Problem 12. Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ that satisfy the following two properties:

(1) $|f(z)| \leq e^{|z|^2}$ for all $z \in \mathbb{C}$,
(2) $f(n^{1/3}) = n$ for all $n \in \mathbb{N}$.

Hint: $f(z) = z^3$ is one of them.