Analysis

Read the instructions of the exam carefully.

Complete this sheet and staple to your answers.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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Pass/Fail recommendation on this form.

Total score: ______________________

Form revised 3/08

Analysis qualifying exam, Spring 2022

Instructions and rubric

- There are 12 problems: 6 on real analysis, 6 on complex analysis.
- Attempt at most five questions on real analysis and five questions on complex analysis. If you submit answers to more questions than this, please indicate clearly which questions should be graded.
- All questions will be graded out of 10 points. Questions with several parts show the breakdown of points in square brackets.
- In case of partial progress on a problem, details will usually earn more points if they are explained as part of a solution outline for the whole problem.

Real analysis

1. (a) [4 points] Given a finite Borel measure μ on $\mathbf{R}$, its **support** is the set

$$S = \{x \in \mathbf{R} : \ \mu((x-\varepsilon, x+\varepsilon)) > 0 \text{ for every } \varepsilon > 0\}.$$

Prove that S is closed, that $\mu(\mathbf{R} \setminus S) = 0$, and that any other set with these two properties must contain S.

(b) [6 points] Prove that there is a finite Borel measure μ on $\mathbf{R}$ such that

(i) μ has support equal to $\mathbf{R}$;

(ii) μ and Lebesgue measure are mutually singular.

2. A function $f : \mathbf{R} \to \mathbf{R}$ is 1**-Lipschitz** if

$$|f(x) - f(y)| \leq |x - y| \qquad \forall x, y \in \mathbf{R}.$$

Let $\mathcal{F}$ be an arbitrary collection of 1-Lipschitz functions, and for each $f \in \mathcal{F}$ let

$$L_f := \{(x, y) \in \mathbf{R}^2 : \ y \leq f(x)\}.$$

Let $L := \bigcup_{f \in \mathcal{F}} L_f$.

(a) [6 points] Prove that L is a Lebesgue measurable subset of the plane. [Hint: L need not be open or closed, but you could try comparing it with certain sets that are.]

(b) [4 points] Prove that L is not necessarily Borel measurable. You may quote without proof the fact that *There exists a non-Borel subset of* $\mathbf{R}$.

3. Let X be a real Banach space and let X^* be its dual. If $Y \subset X$, then let

$$Y^{\perp} := \{\ell \in X^* : \ \ell(y) = 0 \ \forall y \in Y\}.$$

On the other hand, if $Z \subset X^*$, then let

$${}^{\perp}Z := \{x \in X : \ \ell(x) = 0 \ \forall \ell \in Z\}.$$

(a) [5 points] Prove that ${}^{\perp}(Y^{\perp})$ is the closed linear span of Y in X for any $Y \subset X$.

(b) [5 points] Provide an example of a real Banach space X and a subset $Z \subset X^*$ for which $({}^{\perp}Z)^{\perp}$ is not the closed linear span of Z in X^*. [Hint: try something involving the spaces $L^1(m)$, $C([0,1])$ and $L^{\infty}(m)$, where m is Lebesgue measure on $[0,1]$.]

4. Let $f : [0, \infty) \to [0, \infty)$. Assume that $f(0) = 0$ and that f is **convex**, meaning that
$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \quad \forall x, y \ge 0,\ 0 < t < 1.$$
Prove that
$$f(x) = \int_0^x g(y)\, dy$$
for some increasing function $g : [0, \infty) \to [0, \infty)$. [Hint: the question does *not* tell you that f is differentiable, or even continuous.]

5. Let μ be a Borel measure on $\mathbf{R}^2$, and assume it has the following property: for every fixed $r > 0$, the quantity $\mu(B(x, r))$ is finite and independent of x, where $B(x, r)$ is the open ball of radius r around x.

   (a) [5 points] Prove that there is a finite constant c such that $\mu(B(x, r)) \le cr^2$ whenever $0 < r \le 1$.

   (b) [5 points] Prove that μ is a constant multiple of Lebesgue measure.

6. Let $f : \mathbf{R} \to \mathbf{R}$ be smooth and (2π)-periodic. Prove that
$$\int_0^{2\pi} |f'(t)|^2\, dt + \int_0^{2\pi} |f''(t)|^2\, dt \le \int_0^{2\pi} |f(t)|^2\, dt + \int_0^{2\pi} |f'''(t)|^2\, dt.$$

7. Let $f : D(0, 1) \to \mathbf{C} \cup \{\infty\}$ be a meromorphic function on the unit disk $D(0, 1) = \{z \in \mathbf{C} : |z| < 1\}$ that extends continuously to the boundary $\{z \in \mathbf{C} : |z| = 1\}$. Suppose also that $|f(z)| = 1$ whenever $|z| = 1$. Show that f is a rational function, in the sense that there exist polynomials $P, Q : \mathbf{C} \to \mathbf{C}$ with Q not identically zero such that $f(z) = P(z)/Q(z)$ whenever $|z| \le 1$ and $Q(z) \ne 0$.

8. Let U be a connected open subset of $\mathbf{C}$, let V be an open subset of U, and let K be a compact subset of U. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $f : U \to \mathbf{C}$ is a holomorphic function that obeys the bounds $|f(z)| \le \delta$ for all $z \in V$ and $|f(z)| \le 1$ for all $z \in U$, then $|f(z)| \le \varepsilon$ for all $z \in K$. (*Hint:* assume this is not the case, form a sequence of counterexamples, and extract a locally uniformly convergent subsequence.)

9. Establish the identity
$$\int_0^1 \log |1 - re^{2\pi i\theta}|\, d\theta = \max(\log r, 0)$$
for all $0 < r < \infty$.

10. Let n be a natural number, and let α be a complex number with $|\alpha| < 1$. Let $f : \mathbf{C} \to \mathbf{C}$ be the function $f(z) := e^z(1-z)^n - \alpha$.

    (a) [6 points] Show that f has exactly n roots (counting multiplicity) in the right half-plane $\{z : \mathrm{Re} z > 0\}$.

    (b) [4 points] If $\alpha \neq 0$, show that the n roots in (a) are all simple.

11. Let $u : D(0,1) \to \mathbf{R}^+$ be a non-negative harmonic function on the unit disk $D(0,1) := \{z \in \mathbf{C} : |z| < 1\}$ with $u(0) = 1$. Show that

$$\frac{1-|z|}{1+|z|} \leq u(z) \leq \frac{1+|z|}{1-|z|}$$

for all $z \in D(0,1)$.

12. (a) [6 points] Establish the identity

$$\sum_{n \in \mathbf{Z}} \frac{1}{(z-n)^2} = \frac{\pi^2}{\sin^2(\pi z)}$$

for any $z \in \mathbf{C} \backslash \mathbf{Z}$.

    (b) [4 points] Establish the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$