INSTRUCTIONS: Do any 10 of the following questions. If you attempt more than 10 questions, indicate which ones you would like to be considered for credit (otherwise the first 10 will be taken). Each question counts for 10 points. Little or no credit will be given for answers without adequate justification.

#	**Score**	**Counts in 10?**
1	______	___
2	______	___
3	______	___
4	______	___
5	______	___
6	______	___
7	______	___
8	______	___
9	______	___
10	______	___
11	______	___
12	______	___
Total	______	10

Problem 1: Prove that the infinite series

$$f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

converges for all $x\in\mathbb{R}$ and defines a continuous function $f\colon\mathbb{R}\to\mathbb{R}$. Then prove that $f(x)>0$ for all $x\in\mathbb{R}$.

Problem 2: Let $a<b$ be reals and recall that $f\colon[a,b]\to\mathbb{R}$ is Hölder continuous with exponent γ if

$$\sup_{a\le x<y\le b}\frac{|f(y)-f(x)|}{|x-y|^\gamma}<\infty$$

Given $f,g\colon[a,b]\to\mathbb{R}$ that are Hölder continuous with exponent $3/4$, prove that f is Riemann-Stieltjes integrable with respect to g.

Problem 3: Let $a<b$ be reals. Prove that the interval $[a,b]$ is uncountable.

Problem 4: Prove that for each $\lambda\in(0,1]$ there exists a unique continuous $f\colon[0,1]\to[0,\infty)$ that solves the integral equation

$$f(x)=\lambda\int_0^1\mathrm{e}^{-xf(t)}\mathrm{d}t,\quad x\in[0,1]$$

You may need that $\forall a,b\ge0\colon\ |\mathrm{e}^{-a}-\mathrm{e}^{-b}|\le\mathrm{e}^{-\min\{a,b\}}|a-b|$.

Problem 5: Prove that there exists a constant $c\in(0,\infty)$ such that

$$\int_0^1 f(x)^2\mathrm{d}x\le c\int_0^1 f'(x)^2\mathrm{d}x$$

holds for all continuously differentiable $f\colon\mathbb{R}\to\mathbb{R}$ with $f(x+1)=f(x)$ for all $x\in\mathbb{R}$ and $\int_0^1 f(x)\mathrm{d}x=0$.

Problem 6: Let $\{C_n\}_{n\ge1}$ be non-empty, closed subsets of a complete metric space (M,ρ). Assume that these sets are nested, meaning that $C_{n+1}\subseteq C_n$ for each $n\ge1$, and that their diameters

$$\operatorname{diam}(C_n):=\sup\bigl\{\rho(x,y)\colon x,y\in C_n\bigr\}$$

obey $\operatorname{diam}(C_n)\to0$. Prove that $\bigcap_{n\ge1}C_n\ne\emptyset$. Then show by way of an example that the claim may fail without the assumption $\operatorname{diam}(C_n)\to0$.

Problem 7: Let A be an $n\times n$ matrix with column vectors $a_1,a_2,\dots,a_n$. Find the determinant of the matrix with column vectors

$$a_1+a_2,a_2+a_3,\dots,a_{n-1}+a_n,a_n+a_1$$

Problem 8: Let A be a matrix of rank $r > 0$. Show that for any r linearly independent rows and r linearly independent columns of A, the corresponding $r \times r$ minor (i.e., submatrix determinant) is nonzero.

Problem 9: Solve the (system of) ODE $\frac{\mathrm{dx}}{\mathrm{d}t} = A\mathrm{x}$ for a vector-valued function $\mathrm{x}\colon [0,\infty) \to \mathbb{R}^2$ with $\mathrm{x}(0) = (1,1)$ and

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

Problem 10: The Euclidean space $\mathbb{R}^3$ is rotated by $30°$ around the diagonal in the xy-plane. Find the matrix associated with this rotation in the standard Cartesian coordinates. (Treat both the clockwise and the counterclockwise rotations. It is fine to express the answer as a product of matrices.)

Problem 11: Let A, B be two symmetric, positive-definite (and thus invertible) matrices such that $A \leq B$ — which means that $B - A$ is itself positive semi-definite. Prove that then $B^{-1} \leq A^{-1}$. (Do not assume that A and B commute.)

Problem 12: Let V be the vector space of all real-valued polynomials (over field $\mathbb{R}$) of degree at most 2. We endow V with the standard inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)\mathrm{d}x$. Find an orthogonal basis in V containing the polynomial $p(x) = x$.