Geometry/Topology Qualifying Exam

Start each problem on a new sheet of paper.
Write your university identification number at the top of each sheet of paper.

DO NOT WRITE YOUR NAME!
Complete this sheet and staple to your answers.

Read the directions of the exam carefully.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE
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Pass/fail recommend on this form.

Total score: ____________________

Form revised 3/08

QUALIFYING EXAM
Geometry/Topology
March 2022

Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. Let M be a closed (= compact without boundary) $2n$-dimensional manifold and let ω be a closed 2-form on M which is non-degenerate, i.e., for any $p \in M$, the map $T_pM \to T_p^*M$, $X \mapsto i_X\omega(p)$, is an isomorphism. Show that the de Rham cohomology groups $H^{2k}_{dR}(M) \neq 0$ for $0 \leq k \leq n$.

2. Let M be a closed n-dimensional manifold. Let ω be a closed k-form on M, $1 \leq k \leq n$. Prove that for any $p \in M$ there is another closed k-form τ which vanishes on a neighborhood of p and such that $[\omega] = [\tau] \in H^k_{dR}(M)$.

3. Let M be a closed n-dimensional manifold and let Ω be a volume form (i.e., a nonvanishing n-form) on M. Given a vector field X on M, its divergence $\mathrm{div}(X)$ is the smooth function on M defined by the identity:

$$L_X(\Omega) = \mathrm{div}(X)\Omega,$$

where L_X denotes the Lie derivative with respect to X.

(a) (5 pts) Prove that $\int_M \mathrm{div}(X)\Omega = 0$.

(b) (5 pts) Express $\mathrm{div}(X)$ in local coordinates.

4. Let ω be a smooth 1-form on a manifold M and let X and Y be smooth vector fields on M. Use the Cartan formula for Lie derivatives to derive the following formula:

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

5. Let $N \subset \mathbb{R}^n - \{0\}$ be a compact submanifold of dimension m. Show that N is transverse to almost all k-dimensional linear subspaces in $\mathbb{R}^n$. Here "almost all" means the set of subspaces that are not transverse to N has measure zero.

6. Describe all the connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$. Here $\vee$ is the wedge sum.

7. Let X be a CW complex consisting of one vertex p, 2 edges a and b, and two 2-cells f_1 and f_2, where the boundaries of a and b map to p, the boundary of f_1 is mapped to the loop ab^3 (that is first a and then b), and the boundary of f_2 is mapped to ba^3.

(a) (5 pts) Compute the fundamental group $\pi_1(X)$ of X. Is it a finite group?

(b) (5 pts) Compute the homology groups of X with integer coefficients.

8. Let X be a topological space and let $Y = (X \times [0, 1])/\sim$, where $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$ for all $x, x' \in X$. Compute the homology groups of Y in terms of those of X.

9. Let M be a compact odd-dimensional manifold with nonempty boundary ∂M. Show that the Euler characteristics of M and ∂M are related by $\chi(M) = \frac{1}{2}(\partial M)$.

10. Let $A \in GL(n+1, \mathbb{C})$. It induces a smooth map

$$\phi_A : \mathbb{CP}^n \to \mathbb{CP}^n, \quad [(z_0, \ldots, z_n)] \mapsto [A(z_0, \ldots, z_n)],$$

where $[(z_0, \ldots, z_n)]$ is the usual equivalence class of $(z_0, \ldots, z_n)$ in $(\mathbb{C}^{n+1} - \{0\})/(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$, where $\lambda \in \mathbb{C}^\times$. (You do not have to check the smoothness of ϕ_A.)

(a) (3 pts) Show that the fixed points of ϕ_A correspond to eigenvectors of A (up to multiplication by $\mathbb{C}^\times$).

(b) (3 pts) Show that ϕ_A is a Lefschetz map if the eigenvalues of A all have multiplicity 1.

(c) (4 pts) Compute the Euler characteristic of $\mathbb{CP}^n$ by calculating the Lefschetz number of some ϕ_A.