Geometry/Topology Qualifying Exam

Start each problem on a new sheet of paper.
Write your university identification number at the top of each sheet of paper.

DO NOT WRITE YOUR NAME!
Complete this sheet and staple to your answers.

Read the directions of the exam carefully.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE
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Pass/fail recommend on this form.

Total score: ____________________

Form revised 3/08

QUALIFYING EXAM
Geometry/Topology
March 2021

Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. Without using homology groups or homotopy groups, directly derive Brouwer's fixed point theorem (any continuous map $f\colon D^2 \to D^2$ has a fixed point, where D^2 is the closed 2-disk) from the hairy ball theorem (any continuous vector field on S^2 is somewhere 0).

2. Solve the following problems:

(a) Let $F : S^n \to S^n$ be a continuous map. Show that if F has no fixed point, then the degree of the map, $\deg F = (-1)^{n+1}$.

(b) Show that if X has S^{2n} as universal covering space, then $\pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

3. Let $p_1, \ldots, p_n$ be n distinct points in $\mathbb{R}^3$. Calculate the integral homology groups of $\mathbb{R}^3 \setminus \{p_1, \ldots, p_n\}$.

4. Let $\Delta_n^{(k)}$ be the k-dimensional skeleton of the n-simplex Δ_n. Calculate the reduced homology groups $\widetilde{H}_i(\Delta_n^{(k)})$ for all values of i, k, n.

5. Define the complex projective space $\mathbb{CP}^n$ to be the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the relation $x \sim \lambda x$ for all $\lambda \in \mathbb{C} \setminus \{0\}, x \in \mathbb{C}^{n+1} \setminus \{0\}$. Construct a CW complex structure on $\mathbb{CP}^n$ with no odd-dimensional cells and exactly 1 cell in each even dimension up to $2n$. Calculate the fundamental group and the integral homology groups of $\mathbb{CP}^n$.

6. Define the orientation double cover for any topological manifold. What is the orientation double cover of the real projective plane $\mathbb{RP}^n$?

7. Show that $S^2 \times S^2$ and the connected sum $\mathbb{CP}^2 \# \mathbb{CP}^2$ are not homotopy equivalent.

8. Consider a differentiable map $f : S^{2n-1} \to S^n$, with $n \geq 2$. If $\alpha \in \Omega^n(S^n)$ is a differential form of degree n such that $\int_{S^n} \alpha = 1$, let $f^*\alpha \in \Omega^n(S^{2n-1})$ be its pull-back under f.

(a) Show that there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.

(b) Show that the integral $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choices of β and α.

9. Let $f : M \to N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N, respectively, and suppose that $f_*X = Y$ (i.e., $f_*(X(x)) = Y(f(x))$ for all $x \in M$). Then prove that

$$f^*(L_Y \omega) = L_X(f^*\omega)$$

where ω is a 1-form on N. Here L denotes the Lie derivative.

10. Prove Cartan's lemma: Let M be a smooth manifold of dimension n. Fix $1 \leq k \leq n$. Let ω^i and φ_i be 1-forms on M. Suppose that the $\{\omega^1, \ldots, \omega^k\}$ are linearly independent and that $\sum_{i=1}^k \varphi_i \wedge \omega^i = 0$. Prove that there exist smooth functions $h_{ij} = h_{ji} : M \to \mathbb{R}$ such that for all $i = 1, \ldots, k$, $\varphi_i = \sum_{j=1}^k h_{ij}\omega^j$.