Numerical Analysis

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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1. ______________________	5. __________________________
2. ______________________	6. __________________________
3. ______________________	7. __________________________
4. ______________________	8. __________________________

Pass/fail recommend on this form.

Total score: ____________________

Form revised 9/07

Qualifying Exam, Spring 2021

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Let $f(0)$, $f(h)$ and $f(2h)$ be the values of a real valued function at $x = 0$, $x = h$ and $x = 2h$.

(a) Derive the coefficients c_0, c_1 and c_2 so that

$$Df_h(x) = c_0\, f(0) + c_1\, f(h) + c_2\, f(2h)$$

is as accurate an approximation to $f'(0)$ as possible.

(b) Derive the leading term of a truncation error estimate for the formula you derived in **(a).**

[2] (5 Pts.) Let $f(x) = \sqrt{x}$.

(a) Give the linear approximation to $f(x)$ that interpolates this function at $x = 0$ and $x = h$.

(b) Determine the order of the error bound for the linear approximation in (a). Specifically, if $l(x)$ is the linear approximation, find the value α such that there exits a constant C which insures

$$\left\|\sqrt{x} - l(x)\right\| \leq \mathrm{C}\, h^{\alpha} \quad \text{for all } x \in [0, h].$$

[3] (5 Pts.) Given a smooth function $f(x)$ consider the method

$$\int_0^h f(s)ds \approx \frac{h}{2}(f(0) + f(h)) - \frac{h^2}{12}(f'(h) - f'(0))$$

(a) What is the order of accuracy of this method? Justify your answer.

(b) Give the formula for the composite integration formula based on this method, and give the expected order of accuracy of the composite method.

[4] (5 Pts.) Show that when Newton's method is used to determine the root $\bar{x}$ of a smooth function for which $f(\bar{x}) = 0$ and $f'(\bar{x}) = 0$ but $f''(\bar{x}) \neq 0$ the itereration converges linearly. Tip: use expansions of $f(\bar{x} + e^n)$ and $f'(\bar{x} + e^n)$.

[5] (10 Pts.) Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$$

with $y(0) = y_0$ for $t \in [0, \mathrm{T}]$ with $f(y)$ smooth.

(a) Let $h = \dfrac{\mathrm{T}}{\mathrm{N}}$ be the timestep size, derive an error bound for the Taylor series method

$$y^{n+1} = y^n + h\, f(y^n) + \frac{h^2}{2} \frac{\mathrm{d}f}{\mathrm{d}y}(y^n)\, f(y^n)$$

(b) Is the method that is obtained by using the approximation $\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2}|_{t_n} \approx \dfrac{y^{n+1} - 2\, y^n + y^{n-1}}{h^2}$ instead of $\dfrac{\mathrm{d}f}{\mathrm{d}y}(y^n)\, f(y^n)$ in (a) a convergent method? Justify your answer.

[6] (10 Pts.) (a) Construct a second order accurate scheme approximating the scalar differential equation

$$u_t + f(u)_x + g(u)_y = 0$$

for $(x, y) \in [0, 1]x[0, 1]$, $t > 0$ with f, g and $u(x, y, 0)$ smooth functions, and u periodic in x and y with period 1.

(b) For which values of the ratio $\dfrac{dt}{dx}$ does your scheme converge for a small interval $0 < t < \mathrm{T}$? Why, and what prevents convergence for large T?

(c) Construct a first order accurate scheme which converges for all time.

Justify your answers.

[7] (10 Pts.) Consider the initial value problem

$$u_t = u_{xx} \qquad u(x,0) = f(x)$$

for $x \in [0,1]$, $t > 0$, u and f periodic in x with period 1. Describe the expected performance of each of the following schemes:

(a)

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\,dt} = \frac{u_{i+1}^n - 2\,u_i^n + u_{i-1}^n}{dx^2}$$

(b)

$$\frac{u_i^{n+1} - u_i^n}{dt} = \frac{u_{i+1}^{n+1} - 2\,u_i^{n+1} + u_{i-1}^{n+1}}{dx^2}$$

where u_i^n is an approximation to $u(x_i, t_n)$ with $x_i = i\,dx$ and $t_n = n\,dt$.

Discuss stability, convergence and accuracy of each.

[8] (10 Pts.) Consider the problem in two dimensions,

$$\begin{aligned}
-\triangle u + u &= f(x,y), \quad (x,y) \in T, \\
u &= g_1(x), \quad (x,y) \in T_1, \\
u &= g_2(y), \quad (x,y) \in T_2, \\
\frac{\partial u}{\partial n} &= h(x,y), \quad (x,y) \in T_3,
\end{aligned}$$

where

$$\begin{aligned}
T &= \{(x,y)|\ x > 0,\ y > 0,\ x + y < 1\} \\
T_1 &= \{(x,y)|\ y = 0,\ 0 < x < 1\} \\
T_2 &= \{(x,y)|\ x = 0,\ 0 < y < 1\} \\
T_3 &= \{(x,y)|\ x > 0,\ y > 0,\ x + y = 1\}.
\end{aligned}$$

(a) Find the weak variational formulation of the problem and verify the assumptions of the Lax-Milgram Lemma by analyzing the appropriate bilinear and linear forms (impose the weakest necessary assumptions on the functions f, g_1, g_2 and h).

(b) Develop and describe the piecewise linear Galerkin finite element approximation of the problem and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution.