Numerical Analysis

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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1. ______________________
2. ______________________
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Pass/fail recommend on this form.

Total score: ____________________

Form revised 9/07

Qualifying Exam, Spring 2022

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Assume f is a smooth function from $\mathrm{R} \to \mathrm{R}$. For a specified $h > 0$ let $p(x)$ be the polynomial

$$p(x) = f(0) + [\mathrm{D}_0 f](0)\, x + \frac{1}{2}[\mathrm{D}_+\mathrm{D}_- f](0)\, x^2$$

where

$$[\mathrm{D}_0 f](s) = \frac{f(s+h) - f(s-h)}{2h} \qquad [\mathrm{D}_+ f](s) = \frac{f(s+h) - f(s)}{h} \qquad [\mathrm{D}_- f](s) = \frac{f(s) - f(s-h)}{h}$$

(a) For $x \in [-h, h]$ derive the leading term of an error bound for the error, $||f(x) - p(x)||_\infty$. '

(b) At which points does $p(x)$ interpolate $f(x)$?

[2] (5 Pts.) **(a)** Assume $h > 0$ and all values $\alpha_{i\pm\frac{1}{2}} > 0$ for $i = 1 \ldots \mathrm{N}$. Give the $\mathrm{N} \times \mathrm{N}$ matrix representation of the linear difference operator $\mathrm{L} : \mathrm{R}^\mathrm{N} \to \mathrm{R}^\mathrm{N}$ defined by

$$(\mathrm{L}\vec{u})_i = u_i - \gamma \frac{\alpha_{i-\frac{1}{2}}\, u_{i-1} \;-\; (\alpha_{i-\frac{1}{2}} \;+\; \alpha_{i+\frac{1}{2}})\, u_i + \alpha_{i+\frac{1}{2}}\, u_{i+1}}{h^2} \qquad \text{for}\, i = 1 \ldots \mathrm{N}$$

with $u_0 = 0$ and $u_{\mathrm{N}+1} = 0$.

(b) Assuming $\gamma > 0$ is this matrix non-singular? Justify your answer.

[3] (5 Pts.) Consider two points $x_0 = \frac{\pi}{2}$ and $x_1 = \frac{3\pi}{4}$ in $[0, \pi]$. What should c_0 and c_1 be so that the approximation $\int_0^\pi f(x)dx \approx c_0 f(x_0) + c_1 f(x_1)$ is exact for all polynomials of degree less than or equal to 1?

[4] (5 Pts.) **(a)** Every 2×2 non singular matrix A has an LU factorization, A = LU (where L is lower triangular with 1's on the diagonal and U is upper triangular). True or False ? Justify your answer.

(b) Every 2×2 non singular matrix is strictly diagonally dominant. True or False? Justify your answer.

[5] (5 Pts.) Let $\mathrm{F}(y,t)$ be a smooth function from $\mathrm{R}^2 \to \mathrm{R}$ and consider the non-autonomous ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{F}(y,t) \qquad y(0) = y_0 \tag{1}$$

for $t \in [0, \mathrm{T}]$.

(a) Derive the leading term of the local truncation error for the Trapezoidal method

$$y^{n+1} = y^n + \frac{k}{2}\mathrm{F}(y^n, t^n) + \frac{k}{2}\mathrm{F}(y^{n+1}, t^{n+1})$$

applied to (1).

(b) Derive an error bound for the Trapezoidal method in **(a)** applied to (1). State any assumptions on $\mathrm{F}(y,t)$ that are required for your error bound to hold.

(c) Consider the Trapezoidal method in **(a)** applied to the linear time-dependent system of ODE's

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \mathrm{A}(t)\vec{v} \qquad \vec{v}(0) = \vec{v}_0 \tag{2}$$

What must you assume about the matrix $\mathrm{A}(t)$ in order to generalize the derivation of the error bound you derived in (b) to an error bound for the Trapezoidal method applied to (2).

[6] (10 Pts.) Consider the equation

$$u_t = a\, u_{xx} + 2\, b\, u_{xy} + c\, u_{yy}$$

to be solved for $t > 0$, $0 \le x \le 1$, $0 \le y \le 1$, with smooth initial condition $u(x,y,0) = f(x,y)$ and period 1 boundary conditions in x and y,

(a) For what values of the real constants a, b, c is this a well posed problem?

(b) Construct a convergent second order accurate finite difference scheme approximating this initial value problem for these values of a, b, c

Justify your answers

[7] (10 Pts.) Consider the equation

$$u_t + (u(1-u))_x = 0$$

to be solved for $t > 0$, $0 \le x \le 1$ with period 1 boundary conditions and smooth initial data $u(x,0) = f(x)$.

Suppose $0 < f(x) < 1$, construct a convergent finite difference scheme approximating this initial value problem that has the property that the range of the approximate solution values is always contained in [0, 1].

Justify your answer.

[8] (10 Pts.) Develop and describe the piecewise-linear Galerkin finite element approximation of

$$\begin{aligned} -\triangle u + u &= f(x,y), \quad (x,y) \in T, \\ u &= 0, \quad (x,y) \in T_1, \\ u &= 0, \quad (x,y) \in T_2, \\ \frac{\partial u}{\partial n} &= h(x,y), \quad (x,y) \in T_3, \end{aligned}$$

where

$$\begin{aligned} T &= \{(x,y) \mid x > 0,\ y > 0,\ x + y < 1\} \\ T_1 &= \{(x,y) \mid y = 0,\ 0 < x < 1\} \\ T_2 &= \{(x,y) \mid x = 0,\ 0 < y < 1\} \\ T_3 &= \{(x,y) \mid x > 0,\ y > 0,\ x + y = 1\}. \end{aligned}$$

Give a weak variational formulation of the problem. Justify your approximation by analyzing the appropriate bilinear and linear forms (thus show that the corresponding weak variational formulation has a unique solution). Give a convergence estimate and quote the appropriate theorems for convergence.