Qualifying Exam, Spring 2021

Optimization / Numerical Linear Algebra (ONLA)

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

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[1] (10 Pts.) Let $A \in \mathbb{C}^{n\times n}$ have singular value decomposition $A = U\Sigma V^*$.

(a) Prove that A has full rank if and only if A^*A is nonsingular.

(b) Find (with proof) an eigenvalue decomposition for the matrix $B \in \mathbb{C}^{2n\times 2n}$ which is defined by

$$B := \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix},$$

where 0 denotes the $n \times n$ matrix of all zero entries.

[2] **(10 Pts.)** Answer (with proof or counter-example) which of the following sets are convex:

(a) (3 pts.) A hyper-rectangle: $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, n\}$.

(b) (3 pts.) The set of points closer to a given point $z \in \mathbb{R}^n$ than to a given nonempty, closed set $S \subset \mathbb{R}^n$:

$$\{x \in \mathbb{R}^n \mid \|x - z\|_2 \leq \mathbf{dist}(x, S)\},$$

where $\mathbf{dist}(x, S)$ returns the distance between x and S

$$\mathbf{dist}(x, S) = \inf\{\|x - y\|_2 \mid y \in S\}.$$

(c) (4 pts.) The set of points closer to a given nonempty, closed set $S \subset \mathbb{R}^n$ than to a given point $z \in \mathbb{R}^n$:

$$\{x \in \mathbb{R}^n \mid \|x - z\|_2 \geq \mathbf{dist}(x, S)\}.$$

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[3] (10 Pts.) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper, closed, convex function on $\mathbb{R}^n$. Let $\alpha > 0$. Define the proximal operator with respect to αf as

$$\mathrm{Prox}_{\alpha f}(y) = \operatorname*{argmin}_{x \in \mathbb{R}^n} \left\{ \alpha f(x) + \frac{1}{2}\|x - y\|_2^2 \right\}.$$

Given $y \in \mathbb{R}$, the proximal operator returns the unique solution to the minimization problem.

Consider a nonzero vector $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ satisfying $a \neq 0$ and function $g : \mathbb{R}^{mn} \to \mathbb{R}$ defined as

$$g(x^1, \ldots, x^m) = f(a_1 x^1 + \cdots + a_m x^m).$$

Show that, with

$$v = \frac{1}{\|a\|_2^2} \left(a_1 y^1 + \cdots + a_m y^m - \mathrm{Prox}_{\|a\|_2^2 f}(a_1 y^1 + \cdots + a_m y^m) \right),$$

we have

$$\mathrm{Prox}_g(y^1, \ldots, y^m) = \begin{bmatrix} y^1 - a_1 v \\ \vdots \\ y^m - a_m v \end{bmatrix}.$$

(Background: Computing a proximal operator is an optimization problem. When $\mathrm{Prox}_f(y)$ has a closed-form solution (for any input y), it is suitable used as a subroutine. This question shows when $\mathrm{Prox}_f(y)$ has a closed-form solution, Prox_g also has a closed-form solution.

Hint: compare optimality conditions.)

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[4] (10 Pts.)

(a) (3 pts.) Show that if $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite and $Q \in \mathbb{R}^{n \times k}$ has full column rank, then $T = Q^\top A Q$ is also symmetric positive definite.

(b) (7 pts.) Let A and B be real symmetric $n \times n$ matrices. Suppose that $AB = BA$ and that A has distinct eigenvalues. Show that there is an orthogonal matrix Q such that $QAQ^\top$ and $QBQ^\top$ are diagonal matrices.

[5] (10 Pts.)

(a) (5 pts.) Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, $m \geq n$. The normal equations for minimizing $\|Ax - b\|_2^2$ with respect to x are $A^\top Ax = A^\top b$. Derive the normal equations for minimizing $\|Ax - b\|_C^2$, where C is a symmetric positive definite matrix and $\|x\|_C^2 = (x^\top Cx)$.

(b) (5 pts.) Let $A = R + uv^\top$, where R is an upper triangular matrix, and u and v are column vectors. Describe an efficient algorithm to compute the QR decomposition of A. Concretely, provide a pseudocode description of the algorithm and explain why it takes a particular number (asymptotic notation) of operations.
Hint: Using Givens rotations, for $n \times n$ matrices the algorithm should take $O(n^2)$ operations.

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[6] (10 Pts.) Let D be a $m \times m$ diagonal matrix with non-zero diagonal elements $d_1, \ldots, d_m$, and w a vector with non-zero entries $w_1, \ldots, w_m$. Let $A = D + ww^T$ be the diagonal matrix plus a rank-one update. Prove that the eigenvalues λ of A must be roots of the function $f(\lambda) = 1 + \sum_{j=1}^{m} \frac{w_j^2}{d_j - \lambda}$. (Hint: prove A has eigenvector v only if $x = w^T v$ satisfies $x + w^T(D - \lambda I)^{-1} wx = 0$.)

[7] (10 Pts.) Assume $f : \mathbb{R}^n \to \mathbb{R}$ is proper, closed, and convex, , and continuously differentiable. Also assume f is μ-strongly convex and its gradient ∇f is L-Lipschitz. Let $\alpha \in (0, 2/L)$.

Find (with proof) the smallest β in terms of α, μ, L such that

$$\|x' - y'\|_2 \leq \beta \|x - y\|_2 \quad \forall x, y \in \mathrm{dom} f, \quad x' = x - \alpha \nabla f(x), \quad y' = y - \alpha \nabla f(y).$$

(You may make the additional assumption that f is twice continuously differentiable. If you do so and answer this question correctly, you will get 7 pts out of 10. To get all the 10 pts, you must not make this assumption. There are multiple equivalent definitions of μ-strong convexity and L-Lipschitz differentiability. You can use any of those equivalent definitions as long as you clearly state which ones you use.)

[8] **(10 Pts.)** Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Show that the Krylov subspace

$$\mathcal{K}_k(A, b) = \text{span}\{b, Ab, A^2b, \ldots, A^{k-1}b\}$$

has dimension k if and only if the Arnoldi algorithm or the Lanczos algorithm can compute q_k without quitting first.
Hint: The Arnoldi and Lanczos algorithms are shown below.

Arnoldi algorithm for (partial) reduction to Hessenberg form:
$q_1 = b/\|b\|_2$
for $j = 1$ **to** k
  $z = Aq_j$
  for $i = 1$ **to** j
    $h_{i,j} = q_i^\top z$
    $z = z - h_{i,j}q_i$
  end for
  $h_{j+1,j} = \|z\|_2$
  if $h_{j+1,j} = 0$, quit
  $q_{j+1} = z/h_{j+1,j}$
end for

Lanczos algorithm for (partial) reduction to symmetric tridiagonal form:
$q_1 = b/\|b\|_2$, $\beta_0 = 0$, $q_0 = 0$
for $j = 1$ **to** k
  $z = Aq_j$
  $\alpha_j = q_j^\top z$
  $z = z - \alpha_j q_j - \beta_{j-1}q_{j-1}$
  $\beta_j = \|z\|_2$
  if $\beta_j = 0$, quit
  $q_{j+1} = z/\beta_j$
end for