Qualifying Exam, Spring 2022

Optimization / Numerical Linear Algebra (ONLA)

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM. PLEASE USE BLANK PAGES AT END FOR ADDITIONAL SPACE.

1. (10 points) Suppose A and B are square matrices with $\|A - B\|_2 \leq \varepsilon$. Show that if λ is an eigenvalue of A, then $\|(B - \lambda I)^{-1}\|_2 \geq 1/\varepsilon$. (Here I denotes the identity matrix of appropriate size and you may assume $B - \lambda I$ is invertible).

2. (10 points) Prove that if B is a square matrix with $\|B\|_2 < 1$ then $I + B$ is invertible and $\|(I + B)^{-1}\|_2 \leq (1 - \|B\|_2)^{-1}$, where I denotes the identity of appropriate size.

3. (10 points) Suppose x is the solution to $Ax = b$ for an invertible matrix A with condition number $\kappa(A)$. Assume that

$$(A + \nabla A)(x + \nabla x) = b$$

for $\|\nabla A\|_2 < \|A\|/\kappa(A)$. Prove that the conditioning of this linear system satisfies

$$\frac{\|\nabla x\|_2}{\|x\|_2} \le \frac{\|\nabla A\|_2}{\|A\|_2} \cdot \frac{\kappa(A)}{1 - \kappa(A)\frac{\|\nabla A\|_2}{\|A\|_2}}.$$

Hint: You may use the result of question 2.

4. (10 points) Let A be a positive definite symmetric matrix.

   (a) Suppose x^* satisfies the system of equations $Ax^* = b$. Prove that x^* is a minimizer of the quadratic function $f(x) = \frac{1}{2}\langle x, Ax\rangle - \langle x, b\rangle$.

   (b) Given this fact, explain how the method of steepest descent can be used to solve the system $Ax = b$.

5. (10 points) Let $H \in \mathbb{R}^{n \times n}$. Assume that there exists a symmetric positive definite (SPD) matrix $P \in \mathbb{R}^{n \times n}$ such that $B = P - H^T P H$ is SPD. Prove that the iteration

$$x^{(k+1)} = Hx^{(k)} + b, \qquad k = 0, 1, 2, \ldots$$

converges.

6. (10 points) Recall that GMRES for $Ax = b$ minimizes over $\mathcal{K}_i = \langle b, Ab, \ldots, A^{i-1}b \rangle$ in each iteration i.
Let A contain $k \times k$ blocks

$$A = \begin{pmatrix} I & A_1 & & & \\ & I & A_2 & & \\ & & I & \ddots & \\ & & & I & A_{k-1} \\ & & & & I \end{pmatrix},$$

with each block being $n \times n$.

(a) Prove $(I - A)^k = 0$.

(b) Prove that GMRES for $Ax = b$ converges in at most k iterations, independent of n, for any b.

7. (10 points) Recall that $\text{prox}_f(x) = \text{argmin}_z(f(z) + \frac{1}{2}\|z - x\|^2)$. Show the following:

(a) If $f(x) = g(x) + \frac{\rho}{2}\|x - a\|^2$, then

$$\text{prox}_f(x) = \text{prox}_{\frac{1}{1+\rho}g}\left(\frac{1}{1+\rho}x + \frac{\rho}{1+\rho}a\right).$$

(b) If $f(x) = g(ax + b)$ with $a \neq 0$, then

$$\text{prox}_f(x) = \frac{1}{a}(\text{prox}_{a^2 g}(ax + b) - b)$$

(c) If f is closed convex and $f^*(x) = \sup_z\{x^\top z - f(z)\}$ is the convex conjugate of f, then $x = \text{prox}_f(x) + \text{prox}_{f^*}(x)$.
Hint: Try optimality conditions and using the conjugate subgradient theorem, which states that if f is closed convex, then $y \in \partial f(x) \Leftrightarrow x \in \partial f^*(y) \Leftrightarrow x^\top y = f(x) + f^*(y)$.

8. (10 points)

(a) Find all local extremizers of

$$x_1^2 + x_2^3 + 5 \quad \text{subject to} \quad 5 \geq x_2 \geq x_1^2.$$

Fully justify your answer using KKT conditions. For orientation, draw a sketch of the level sets of the objective function, the constraints, and the feasible set.

(b) Consider the problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & x^T W x \\ \text{subject to} \quad & x_i^2 = 1, \quad i = 1, \ldots, n, \end{aligned}$$

where W is a symmetric $n \times n$ matrix. Write the Lagrangian for this problem and compute the Lagrange dual function. Provide the best lower bound that you can on the optimal value of the original problem based on the Lagrange dual. Justify your answer!