Algebra Qualifying Exam

Read the instructions of the exam carefully.

Complete this sheet and staple to your answers.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW

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Total score: ____________________

Pass/fail recommend on this form.

Revised 3/30/2010

ALGEBRA QUALIFYING EXAM

2021 MARCH

All answers must be justified. State clearly any theorem that you use.

Problem 1. Prove that the direct sum $\coprod_p \mathbb{Z}/p\mathbb{Z}$ over all prime integers p is not a direct summand of the product $\prod_p \mathbb{Z}/p\mathbb{Z}$.

Problem 2. Let $P \subset \mathbb{Z}[X]$ be a prime ideal such that $\mathbb{Z} \cap P = 0$. Prove that P is a principal ideal.

Problem 3. Prove that every group generated by two involutions (elements of order 2) is solvable.

Problem 4. Prove that the field extension $\mathbb{Q}(\sqrt{-3} + \sqrt[6]{2})$ over $\mathbb{Q}$ is Galois and determine its Galois group.

Problem 5. Let G be a finite group and let $g \in G$. Suppose for every irreducible complex character χ of G we have $|\chi(g)| = |\chi(1)|$. Prove that g is in the center of G.

Problem 6. Let A be a commutative ring, let P be a flat A-module and let I be an injective A-module. Show that $\mathrm{Hom}_A(P, I)$ is an injective A-module.

Problem 7. Let p be a prime number, k a field of characteristic p and G be a (finite) p-group. Let M be a finitely generated kG-module that admits a k-basis B such that $G \cdot B \subseteq B \cup -B$ (*i.e.* $\forall g \in G, \forall\, b \in B$, we have $g \cdot b = \pm b'$ for $b' \in B$). Show that M admits a k-basis B' invariant under G (*i.e.* $G \cdot B' \subseteq B'$ without sign).

Problem 8. Let A be a (non-zero) ring in which the only right ideals are (0) and A. Show that A is a division ring.

Problem 9. Let R be a commutative ring and A, B be two (not necessarily commutative) R-algebras. Consider the functor $\mathrm{Hom}_{R\text{-}\mathsf{Alg}}(A \otimes_R B, -)\colon R\text{-}\mathsf{Alg} \to \mathsf{Sets}$, from R-algebras to sets. Construct two homomorphisms $f\colon A \to A \otimes_R B$ and $g\colon B \to A \otimes_R B$ and show that they induce an injection

$$\eta_C\colon \mathrm{Hom}_{R\text{-}\mathsf{Alg}}(A \otimes_R B, C) \longrightarrow \mathrm{Hom}_{R\text{-}\mathsf{Alg}}(A, C) \times \mathrm{Hom}_{R\text{-}\mathsf{Alg}}(B, C)$$

natural in $C \in R\text{-}\mathsf{Alg}$. Identify the image of η_C explicitly.

Problem 10. Let A be a ring. Let $m, n \geq 1$ and P be a right A-module such that $P^n \simeq A^m$. Show that $S \mapsto P \otimes_A S$ defines a bijection between the set of isomorphism classes of simple A-modules and that of simple $\mathrm{End}_A(P)$-modules.