Student ID Number: ______________________________

UCLA MATHEMATICS – BASIC EXAM: SPRING 2022

INSTRUCTIONS: Do any 10 of the following questions. If you attempt more than 10 questions, indicate which ones you would like to be considered for credit (otherwise the first 10 will be taken). Each question counts for 10 points. Little or no credit will be given for answers without adequate justification. You have 4 hours. Good luck.

NOTATION: We denote by $\mathbb{N} = 1, 2 \dots$ the natural numbers, and by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers, respectively.

#	Score	Counts in 10?
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
Total		10

1. (i) Let $A = (a_{i,j})$ be an $n \times n$ matrix given by

$$a_{i,j} = \begin{cases} 1, & \text{if } i \text{ divides } j; \\ 0, & \text{otherwise.} \end{cases}$$

Find the determinant of A.

(ii) Let $B = (b_{i,j})$ be an $n \times n$ matrix such that $b_{i,j}$ is the number of common divisors of i and j. Find the determinant of B.

2. Consider the Euclidean space $\mathbb{R}^n$ endowed with the Euclidean norm. Define what it means for a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ to be *orthogonal*. Then prove that for any $x, y \in \mathbb{R}^n$ of the same Euclidean norm, there exists an orthogonal map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(x) = y$ and $T(y) = x$.

3. Let A be an $n \times n$ matrix of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Find the Jordan normal form of A^2. Note that n is an arbitrary positive integer.

4. Let A be a complex $n \times n$ matrix such that $A^2 = I_n$, where I_n is the $n \times n$ identity matrix. Show that

$$\operatorname{rank}(A + I_n) + \operatorname{rank}(A - I_n) = n.$$

5. Let $a < b$ be real numbers and $f_1, \dots, f_n \colon [a,b] \to \mathbb{R}$ continuous functions. Define an $n \times n$-matrix $M = (m_{ij})_{i,j=1,\dots,n}$ by

$$m_{ij} := \int_a^b f_i(x) f_j(x) \, \mathrm{d}x.$$

Prove that

$$\det(M) = 0 \quad \iff \quad f_1, \dots, f_n \text{ are linearly dependent.}$$

6. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that

$$a_0 = 1, \quad a_1 = 1, \quad \text{and} \quad a_{n+2} = a_{n+1} + 2a_n \quad \text{for all } n \geq 0.$$

Compute a_n as a function of n.

7. Let $f\colon (0,\infty) \to \mathbb{R}$ be continuous such that $f(4x) = f(x)$ for all x. Prove that there exists $x > 0$ such that $f(x) = f(2x)$.

8. Let $f : [0, 2\pi] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{N\to\infty} \int_0^{2\pi} f(x)\sin(Nx)\,\mathrm{d}x = 0.$$

9. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuously differentiable function with $\mathrm{grad} f \neq 0$ at 0. Show that there are two other continuously differentiable functions $g : \mathbb{R}^3 \to \mathbb{R}$ and $h : \mathbb{R}^3 \to \mathbb{R}$ such that the function

$$(x, y, z) \to (f(x, y, z), g(x, y, z), h(x, y, z))$$

from $\mathbb{R}^3$ to $\mathbb{R}^3$ is one-to-one in some neighborhood of 0.

10. Assume that K is a closed subset of a complete metric space (X, d) with the property that, for any $\epsilon > 0$, K can be covered by a finite number of sets of the form $B_\epsilon(x)$, where

$$B_\epsilon(x) := \{y \in X : d(x, y) < \epsilon\}.$$

Prove that K is compact. Justify all claims you make.

11. Let $(a_n)_{n=0}^{\infty}$ be a sequence of positive reals such that

$$a = \lim_{n\to\infty} na_n$$

exists with $a \in (0, \infty)$. Prove that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent with $f(x)$ continuously differentiable for all $x \in (-1, 1)$. Then show

$$\lim_{x\to 1^-} f'(x)(1-x) = a.$$

12. Let $a < b$ be reals. Prove that if a bounded function $f\colon (a,b) \to \mathbb{R}$ is uniformly continuous then the limits

$$\lim_{x\to a^+} f(x) \quad \text{and} \quad \lim_{x\to b^-} f(x)$$

exist. Justify all claims you make.