Chapter 1
Closed-form Analytic Solutions of the Problem of a Hollow Sphere Made of Second Gradient Plastic Porous Material and Subjected to Hydrostatic Loading

R. Burson and K. Enakousta

Abstract Gologanu, Leblond, Perrin, and Devaux (GLPD) developed a constitutive model for ductile fracture for porous metals based on generalized continuum mechanics assumptions. The model predicted accurately ductile fracture process in porous metallic structures under several complex loads. The GLDP model's performances over its competitors has attracted the attention of several authors who explored further capabilities of the model. The aim of this paper is to provide analytical solutions for the problem of a porous hollow sphere subjected to hydrostatic loadings, the matrix of the hollow sphere obeying the GLPD model. The exact solution for the expressions of the stress and the generalized stress the GLPD model involved are illustrated for the case where the matrix material does not contain any voids. The results show that the singularities obtained in the stress distribution with the local Gurson model are smoothed out, as expected with any generalized continuum models. The paper also presents some elements of the analytical solution for the case where the matrix is porous and obeys the full GLPD model at the initial time when the porosity is fixed. These analytical solutions can serve as benchmark solutions to assess numerical implementations of any second gradient constitutive model.

Keywords: Gradient Model · Analytical Solutions · Plasticity · Hollow Sphere Problem · Fracture

---

R. Burson and
Department of Mathematics, California State University, Northridge, 18111 Nordhoff Street, Northridge, CA 91330
e-mail: Roy.burson.618@my.csun.edu

K. Enakousta
Department of Mathematics, California State University, Northridge, 18111 Nordhoff Street, Northridge, CA 91330 and Department of Mathematics, UCLA, Los Angeles, 520 Portola Plaza, Los Angeles, CA, 90095

1.1 Introduction

Metal structures often fail by ductile rupture when they are subjected to external static or dynamic forces. The requirement to develop constitutive models (whether they are physics, mechanics and/or mathematics based) that can predict precisely ductile fracture processes in metals has become a key point in metal structure design community. So far, such community has widely accepted that the model proposed by [17] and extended by [29, 30] to account for cavities interactions and coalescence (these features were disregarded in [17]'s original work) following an earlier suggestion by [24] can adequately describes ductile fracture in metals. Several extensions have followed these pioneering works; among them, let us mention the contributions of [20], [19], and recently [23]. The latter has modified Gurson model to include shear failure which often occurs, for instance, during high velocity impact failures of many steel materials.

Another modification of Gurson model including a characteristic length scale aimed at eliminating the pathological post-bifurcation mesh dependence issues proposed by [18] based on a previous suggestion of [21] in the context of concrete damage was adopted by [26, 25]. This proposal was studied in details by [7, 10] and adopted (thanks to its successes) in the context of high rate deformation and failure of materials by [11], [12], and [1]. However, the proposal was of less satisfaction from a theoretical and physical view points since it does not rely on any serious physical justification. This was the motivation of the development by Gologanu, Leblond, Perrin, and Devaux [16] of a second-gradient micromorphic model[1] for porous plastic materials. The GLPD model was obtained from a refinement of [17]'s original homogenization procedure, which was based on conditions of homogeneous boundary strain rate.

In contrast, the boundary velocity in the GLPD model approach was assumed to be a quadratic, rather than linear, function of the coordinates. The physical idea behind this assumption was to account in this way for possible quick variations of the macroscopic strain rate over very short distances, for example at the scale of the elementary cell the GLPD model is based on. The output of the procedure was a model of "micromorphic" nature, involving the second gradient of the macroscopic velocity and generalized macroscopic stresses of "moment" type (homogeneous to the product of a stress and a distance.) Other type of higher-order gradient models involing third-rank stress tensor with applications in bone remodeling design and other domain of interest exist. Among them let us mention the works by [28, 6, 4].

[1] micromorphic model will simply be denoted by GLPD model for shortness.

In practice, the GLPD model was extensively studied by [7, 15] who have notably shown that the model has the ability to predict mesh-independent FE solutions and to reproduce satisfactorily ductile fracture tests. Other numerical simulations involving second gradient models are available in the literature, see for instance [2, 27, 22]. A recent modification of the GLPD model numerical implementation developed in [7, 15] was suggested by [3] and yielded the same conclusions. The assessment of the reliability and accuracy of these two algorithms requires the development of analytical solutions that have served as critical cross references, see [7, 8, 14]. These solutions are based on two crude approximations so as for analytical solutions to be amenable: the porosity in the matrix material of the geometry considered was assumed to vanish.

The objective of the present paper is to follow up the study of applications of the GLPD model to simple problems that might be of interest to validate the numerical implementation of this model into a finite element code. The problem considered here is a hollow sphere subjected to a hydrostatic tension and made of porous plastic material, obeying the GLPD model. We found the analytic solution of the hollow sphere problem in terms of deformation, stress and moment distributions under the conditions that the matrix obeys a reduced GLPD model for the case where the porosity vanishes. We also consider some elements of solution of the problem in the presence of porosity in the matrix material, which is a rather complex type of problem. The complexity of the latter problem (a highly non-linear type of problem) forces us to present only some elements of the analytical solution at the initial time when the porosity is held constant. The rest of the paper is structured as follows.

- The first section describes the problem model, whereas the next section presents the details of the analytical derivation of the exact solution for the case where the porosity vanishes as in [9].

- The follow up section assesses the solution obtained for the case where the porosity vanishes. An algorithm that simulates the behavior of his model and analyzes the effects of the characteristic length scale on the distribution of stress and moments is also presented.

- Finally, we considered the solution of the problem for the case where the porosity does not vanish at the initial time. We provide implicit analytic expressions for the Cauchy stress and moment components based on a highly non linear ordinary differential equation, which involves the characterstic length scale of the GLPD model.

1.2 Description of the Hollow Sphere Problem

We consider a hollow sphere of inner radius r_i, outer radius r_e, representing an elementary cell of a porous plastic metal, see Figure 1.1. The boundary of the central void is free of traction whereas the outer boundary is subjected to some overall hydrostatic tension $\boldsymbol{T}$. The details of the derivation of these boundary conditions can be found in [8]. The matrix material of the porous hollow sphere is supposed to obey the GLPD constitutive model.

The hollow sphere model problem presented here have served to find the solutions of several ductile fracture problems the solution of which have yields micromechanics based models for ductile porous metals under various loading conditions. Some of these problems as well as their solutions can be found in the works of [19], [20], and [7, 8] to mention a few of them.

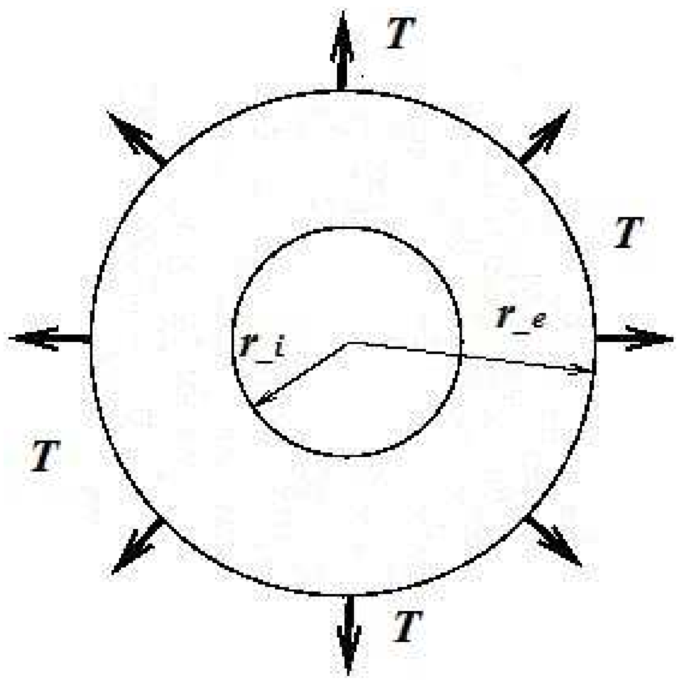


Fig. 1.1 An illustration of the hollow sphere model problem

1.3 Solution to the Hollow Sphere Problem When the Porosity is Neglected

We are seeking a solution of the spherical shell problem for purely ideal-plastic behavior, the yield stress in simple tension being denoted by Σ_0 and the porosity in the matrix neglected. As a result, the yield criterion Eq.(1.47) reduces

$$\Phi(\boldsymbol{\Sigma}, \boldsymbol{M}, \Sigma) \equiv \frac{1}{\bar{\Sigma}^2}\left(\Sigma_{eq}^2 + \frac{Q^2}{b^2}\right) - 1 = 0. \tag{1.1}$$

. In this equation $\boldsymbol{\Sigma}$ represents the ordinary second-rank symmetric Cauchy stress tensor and $\boldsymbol{M}$ is the third-rank "moment tensor" symmetric in its first two indices only. The components of $\boldsymbol{M}$ are related through the conditions.

$$M_{ijj} = 0. \tag{1.2}$$

In the same expression:

- $\Sigma_{eq} \equiv \left(\frac{3}{2}\boldsymbol{\Sigma}' : \boldsymbol{\Sigma}'\right)^{1/2}$ ($\boldsymbol{\Sigma}'$: deviator of $\boldsymbol{\Sigma}$) is the von Mises equivalent stress.
- Σ represents a kind of average value of the yield stress in the heterogeneous metallic matrix.
- Q^2 is a quadratic form of the components of the moment tensor given by

$$Q^2 \equiv A_1\mathcal{M}_1 + A_2\mathcal{M}_2 \quad , \quad \begin{cases} A_1 = 0.194 \\ A_2 = 6.108 \end{cases} \tag{1.3}$$

  where $\mathcal{M}_1$ and $\mathcal{M}_2$ are the quadratic invariants of $\mathbf{M}$ defined by:

$$\begin{cases} \mathcal{M}_1 \equiv M_{mi}M_{mi} \\ \mathcal{M}_2 \equiv \frac{3}{2}M'_{ijk}M'_{ijk}, \end{cases} \tag{1.4}$$

  $M_{mi} \equiv \frac{1}{3}M_{hhi}$ and $\mathbf{M}'$ denoting the mean and deviatoric parts of $\mathbf{M}$, taken over its first two indices.
- b represents the characteristic lenght scale.

The flow rule, Eq.(1.51), becomes, after development (see [7, 15] for the details),

$$\begin{cases} \dot{D}_{ij}^{p} = \eta \dfrac{3}{\Sigma_0^2}\,\Sigma'_{ij} \\ \left(\nabla\dot{D}\right)_{ijk}^{p} = \dfrac{\eta}{{\Sigma_0}^2 b^2}\left(\dfrac{2}{3}A_I\delta_{ij}M_{mk} + 3A_{II}M'_{ijk}\right) + \delta_{ik}U_j + \delta_{jk}U_i, \end{cases} \tag{1.5}$$

$M_{mk} \equiv \frac{1}{3}M_{hhk}$ and $\boldsymbol{M}'$ denoting the mean and deviatoric parts of $\boldsymbol{M}$, taken over its first two indices; η is the plastic multiplier, determined from the consistency condition and defined as

$$\eta = \begin{Bmatrix} = 0 & \text{if} & \Phi(\boldsymbol{\Sigma}, \mathbf{M}, \Sigma) < 0 \\ \geq 0 & \text{if} & \Phi(\boldsymbol{\Sigma}, \mathbf{M}, \Sigma) = 0. \end{Bmatrix}$$

We shall also assume that the parameter A_I=0 for the analytical solution to be amenable. Another, more elaborate reason for this choice is that the value of A_I in the GLPD model, 0.194, is very small with respect to that of A_{II}, 6.108; hence, the value of A_I can safely be neglected.

We are looking for a solution in which the spherical shell is entirely plastic, so that the yield function $\Phi(\boldsymbol{\Sigma}, \boldsymbol{M}, \Sigma_0, f)$ is zero everywhere. Since such a solution was already presented in [9], only a summary of the procedure is given in this work.

Let us consider the velocity, strain rate and its gradient fields first. As in the case of purely elastic behavior, the matrix of spherical shell is incompressible; as a result, the velocity field is radial and given by

$$\mathcal{U} = \frac{A}{r^2}, \tag{1.6}$$

where A is a parameter which is independent of the material point position r. (In fact, the expression of the velocity field $\mathcal{U}$ is obtained by writing the constraint that $\mathrm{d}iv(\mathcal{U})$ equals zero because of the assumption of incompressibility of the matrix of the spherical shell.) Following this definition, the non-zero components of the strain rate are found as

$$D_{rr} = -\frac{2A}{r^3}, \quad D_{\theta\theta} = D_{\phi\phi} = \frac{A}{r^3}. \tag{1.7}$$

Thanks to the spherical symmetries involved in the problem, the components of the gradient of the strain rate are defined, here also, as

$$\begin{cases} (\nabla D)_{rrr} \\ (\nabla D)_{\theta\theta r} = (\nabla D)_{\phi\phi r} \\ (\nabla D)_{r\theta\theta} = (\nabla D)_{r\phi\phi} \\ other \quad (\nabla D)_{ijk} = 0. \end{cases} \tag{1.8}$$

The resulting strain gradient components are defined as in [13].

The non-zero components of the stress and moment fields are found using the flow rule, Eq.(1.5), and the incompressibility of the material (which yields $\mathbf{U}$=0). These components are obtained as

$$\begin{cases} \Sigma'_{rr} = \dfrac{1}{\eta}\left(-\dfrac{2A\Sigma_0^2}{3r^2}\right) \\ \\ \Sigma'_{\theta\theta} = \Sigma'_{\phi\phi} = \dfrac{1}{\eta}\left(\dfrac{A\Sigma_0^2}{3r^2}\right) \end{cases} \tag{1.9}$$

and

$$\begin{cases} M'_{rrr} = \dfrac{1}{\eta}\dfrac{2A\Sigma_0^2 b^2}{A_{II} r^4} \\ \\ M'_{\theta\theta r} = M'_{\phi\phi r} = -\dfrac{1}{\eta}\dfrac{A\Sigma_0^2 b^2}{A_{II} r^4} \\ \\ M_{r\theta\theta} = M_{r\phi\phi} = -\dfrac{1}{\eta}\dfrac{A\Sigma_0^2 b^2}{A_{II} r^4} \end{cases} \tag{1.10}$$

The conditions Eq.(1.40) and a combination of the definitions of M'_{rrr} and $M_{r\theta\theta}$ given by the relations Eq.$(1.10)_1$ and Eq.$(1.10)_3$ yield

$$\begin{cases} M_{rrr} = -2M_{r\theta\theta} \\ \\ M_{\theta\theta r} = M'_{\theta\theta r}. \end{cases} \tag{1.11}$$

Replacing the formulas for the stress Eq.(1.9) and moment Eq.(1.10) in the reduced yield criterion, Eq.(1.1), we get the following expression for the plastic multiplier η:

$$\eta = \frac{A\Sigma_0}{r^3}\sqrt{1 + \frac{15}{A_{II}}\frac{b^2}{r^2}}. \tag{1.12}$$

The explicit relation of the plastic multiplier Eq.(1.12) completes the definition of the non-zero components of the moment tensor. However, the full expressions of the non-zero components of the ordinary stress tensor are still unknown. After a tedious but straightforward calculation which uses (i) the expressions of the non-zero components of the moment tensor, (ii) the spherical symmetry properties of the problem, and (iii) the fact that $\Sigma_{rr} - \Sigma_{\theta\theta} = \Sigma'_{rr} - \Sigma'_{\theta\theta}$, the formulas for the non-zero components of the ordinary Cauchy stress tensor are obtained as

$$\frac{d\Sigma_{rr}}{dr} = f(r) \tag{1.13}$$

with

$$\begin{cases} f(r) = \dfrac{2A\Sigma_0^2}{\eta r^3} + \dfrac{2(\eta''\eta^2 - 2\eta'^2\eta)}{\eta^4}\dfrac{A\Sigma_0^2 b^2}{A_{II} r^4} - \dfrac{28\eta'}{\eta^2}\dfrac{A\Sigma_0^2 b^2}{A_{II} r^5} \\ -\left(\dfrac{72}{\eta} + \dfrac{2\eta'}{\eta^2}\right)\dfrac{A\Sigma_0^2 b^2}{A_{II} r^6} - \dfrac{8A\Sigma_0^2 b^2}{\eta A_{II} r^7} \end{cases} \tag{1.14}$$

where η' and η'' denote the first and second derivatives of the plastic multiplier η with respect to r. Eq.(1.13) implicitly defines the expression of the component Σ_{rr} of the stress tensor. The non-zero components of the stress tensor are obtained as

$$\Sigma_{rr} = \int_{r_i}^{r} f(\tau)d\tau; \quad \Sigma_{\theta\theta} = \Sigma_{\phi\phi} = \Sigma_{rr} - \frac{1}{\eta}\left(\frac{A\Sigma_0^2}{r^2}\right). \tag{1.15}$$

The solution of Eq.(1.15) along with the non-zero components of the moment provided above automatically satisfy the balance equations.

Quantity	Symbol	Unit	Value
internal radius	r_i	m	0.05
yield stress	Σ_0	MPa	100
parameter 1	A	m	0.001
parameter 2	A_1	-	0.194
parameter 3	A_2	-	6.108

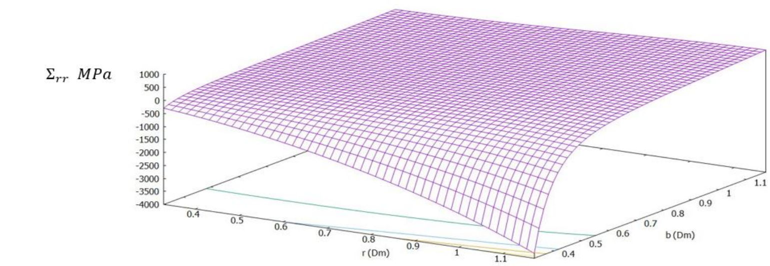
Table 1.1 List of constants parameters

1.4 Numerical Illustrations of the Solution

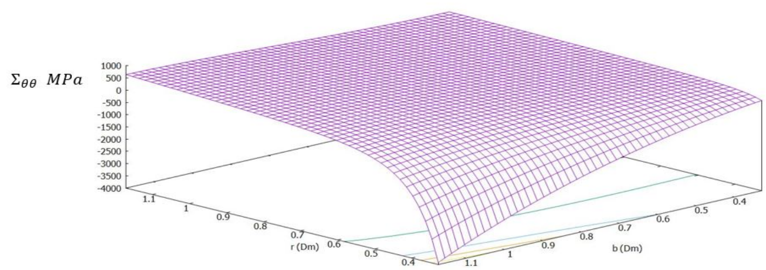
The purpose of this section is to illustrate analytical solution presented in [13]. More specifically, we shall derive the explicit expressions for the stress and moment tensors left off from [13]'s findings. We do so by evaluating the integral Eq.(1.15) using a FORTRAN routine we developed. Each integral has also been evaluated analytically so that we possess the exact solution. The explicit solution to each integral is provided in Appendix § 1.8. We evaluate the following integrals:

$$\int_{r_i}^{r_e} \frac{2A\Sigma_0^2}{\eta r^3} dr$$

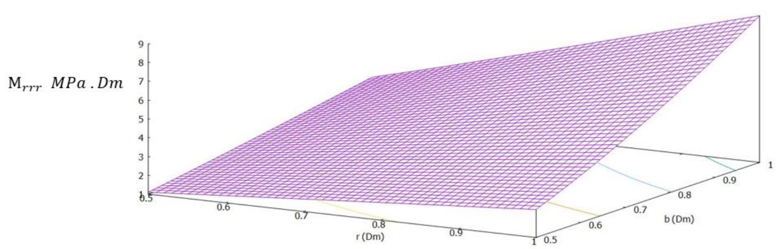
$$\int_{r_i}^{r_e} \frac{2(\eta''\eta - \eta'^2)}{\eta^2}\left(\frac{A\Sigma_0^2 b^2}{A_2 r^4}\right) dr$$



(a) Stress component Σ_{rr}



(b) Stress component $\Sigma_{\theta\theta}$



(c) Moment component M_{rrr}

Fig. 1.2 Illustration of the stress components Σ_{rr}, $\Sigma_{\theta\theta}$ and moment component M_{rrr}.

$$\int_{r_i}^{r_e} \frac{20\eta' + 8\eta'\eta}{\eta^2}(\frac{A\Sigma_0^2 b^2}{A_2 r^5})dr$$

$$\int_{r_i}^{r_e} \frac{(72 + 2\eta')}{\eta}(\frac{A\Sigma_0^2 b^2}{A_2 r^6})dr$$

$$\int_{r_i}^{r_e} \frac{A\Sigma_0^2 b^2}{\eta A_2 r^7}dr$$

Table 1.1 identifies the material and model parameters that are used to obtain the results mentioned in this work. Figures (a), (b), and (c) illustrate the solutions for the stress components Σ_{rr}, $\Sigma_{\theta\theta}$, and the moment components M_{rrr} and $M_{r\theta\theta}$ respectfully.

Figure 1.3 illustrate the analytical expressions of the non zero components of the Cauchy stress and the moment tensors as obtained in [9]. The figure shows that singularities are absent from the stress. Also, there is no discontinuity near the void as the first gradient Gurson model would have predicted.

1.5 Analytic Results in the Presence of Porosity in the Matrix

In this section we present the solution to the hollow sphere problem for the case where the matrix obeys the full GLPD model described in the Appendix § 1.8. Therefore, we loose the simplification that yields the prior results [9]. Most of the results, if not all of them, presented in the works [19, 20, 7, 8] assume incompressibility within the matrix of the material. This being said a complete solution of the hollow sphere problem obeying the GLPD model has not been provided yet.

1.5.1 Derivation of Cauchy Stress Components

The setup of the problem is provided by the balance equation of the problem which are provided in [9] (and are not duplicated here), the yield criterion, and the booundary conditions which are also given in [9]:

$$\begin{cases} \dfrac{d\Sigma_{rr}}{dr} + \dfrac{2}{r}(\Sigma_{rr} - \Sigma_{\theta\theta}) - \dfrac{d^2 M_{rrr}}{dr^2} - \dfrac{4}{r}\dfrac{dM_{rrr}}{dr} - \dfrac{2}{r^2}M_{rrr} \\ \qquad + \dfrac{2}{r^2}\dfrac{dM_{\theta\theta r}}{dr} + \dfrac{4}{r}\dfrac{dM_{r\theta\theta}}{dr} + \dfrac{8}{r^2}M_{r\theta\theta} = 0 & (1.16.a) \\ \frac{1}{\Sigma^2}\left(\Sigma_{eq}^2 + \frac{Q^2}{b^2}\right) + 2p\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma}\right) - 1 - p^2 = 0, & (1.16.b) \end{cases}$$

with M_{rrr} and $M_{r\theta\theta}$ the components of the moment tensor $\mathbf{M}$, Σ_{rr} and Σ_{θ} the non zero components of the stress tensor $\mathbf{\Sigma}$, Σ_m the mean stress, Σ_{eq} the Von Mises stress. The boundary conditions are

$$\begin{cases} r^2\Sigma_{rr} - \frac{d\left(r^2 M_{rrr}\right)}{dr} + 4rM_{r\theta\theta} = 0, \\ \\ M_{rrr} \qquad\qquad\qquad\qquad\quad = 0 \end{cases}$$

for $r = r_i$ and $r = r_e$. The stress components Σ_{rr} and $\Sigma_{\theta\theta}$ in the balance equation Eq.(1.16.a) can be expressed in terms of the invariants Σ_m and Σ_{eq} as followed (see Appendix § 1.9)

$$\frac{d\Sigma_{rr}}{dr} + \frac{2}{r}(\Sigma_{rr} - \Sigma_{\theta\theta}) = \frac{2}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq} + \frac{2}{r\sqrt{3}}\Sigma_{eq} + \frac{d}{dr}\Sigma_m \tag{1.17}$$

Upon substituting Eq.(1.17) into Eq.(1.16.a) and using the yield criterion Eq.(1.16.b) we then find the differential equation

$$\begin{aligned} \frac{d}{dr}\Sigma_{eq}\left(\sqrt{\Sigma_0 - \Sigma_{eq}^2} - 2\Sigma_{eq}\right) + \frac{2\alpha}{r}\Sigma_{eq}\left(\sqrt{\Sigma_0 - \Sigma_{eq}^2}\right) - \alpha\left(\Sigma_0 - \Sigma_{eq}^2\right) \\ = \sinh\left(\gamma\Sigma_m\right)\left(\rho\frac{d}{dr}\Sigma_m - 2p\right) \end{aligned} \tag{1.18}$$

with $\alpha = 3\sqrt{3}$, $\gamma = \frac{3}{2\Sigma_0}$, and $\rho = p\Sigma_0$. Since the invariants Σ_{eq} and Σ_m are independent of one another both sides of Eq.(1.18) must be equal to some constant value, say λ, so then

$$\frac{d}{dr}\Sigma_{eq}\left(\sqrt{\Sigma_0 - \Sigma_{eq}^2} - \beta\Sigma_{eq}\right) + \frac{2\alpha}{r}\Sigma_{eq}\left(\sqrt{\Sigma_0 - \Sigma_{eq}^2}\right) - \alpha\left(\Sigma_0 - \Sigma_{eq}^2\right) = \lambda$$

and

$$\rho\sinh\left(\gamma\Sigma_m\right)\frac{d}{dr}\Sigma_m - 2p\sinh\left(\gamma\Sigma_m\right) = \lambda.$$

The value λ is to be determined later. We found that the solution for Σ_m satisfies the implicit expression

$$\frac{\rho\left(\frac{\Sigma_m}{p} - \frac{\lambda\ln\left(\frac{\left|2pe^{-\gamma\Sigma_m} - \sqrt{4p^2+\lambda^2} - \lambda\right|}{\left|2pe^{-\gamma\Sigma_m} + \sqrt{4p^2+\lambda^2} - \lambda\right|}\right)}{\gamma p\sqrt{4p^2+\lambda^2}}\right)}{2} = r + C \tag{1.19}$$

for some arbitrary constant C which is to be determined from the boundary condition (see Appendix § 1.9). The formulation for Σ_{eq} is determined by the differential equation

$$\frac{d}{dr}\Sigma_{eq} = \frac{\alpha(\Sigma_0 - \Sigma_{eq}^2) - \frac{6}{r}\Sigma_{eq}\left(\sqrt{\Sigma_0 - \Sigma_{eq}^2}\right) + \lambda}{\sqrt{\Sigma_0 - \Sigma_{eq}^2} - \beta\Sigma_{eq}} \tag{1.20}$$

The constant C is provided by the formula

$$C = \frac{\rho\left(\frac{\Sigma_m(r_i)}{p} - \frac{\lambda\ln\left(\frac{\left|2pe^{-\gamma\Sigma_m(r_i)} - \sqrt{4p^2+\lambda^2} - \lambda\right|}{\left|2pe^{-\gamma\Sigma_m(r_i)} + \sqrt{4p^2+\lambda^2} - \lambda\right|}\right)}{\gamma p\sqrt{4p^2+\lambda^2}}\right)}{2} - r_i \tag{1.21}$$

The constant λ is found by solving the root to the equation

$$\left(\frac{\Sigma_m(r_e)-\Sigma_m(r_i)}{p} - \frac{\lambda\ln\left(\frac{\left|2pe^{-\gamma\Sigma_m(r_e)}-\sqrt{4p^2+\lambda^2}-\lambda\right|}{\left|2pe^{-\gamma\Sigma_m(r_e)}+\sqrt{4p^2+\lambda^2}-\lambda\right|}\cdot\frac{\left|2pe^{-\gamma\Sigma_m(r_i)}-\sqrt{4p^2+\lambda^2}-\lambda\right|}{\left|2pe^{-\gamma\Sigma_m(r_i)}+\sqrt{4p^2+\lambda^2}-\lambda\right|}\right)}{\gamma p\sqrt{4p^2+\lambda^2}}\right) - \frac{2\left(r_i - r_e\right)}{\rho} = 0$$

From here we completely solve for Σ_{eq} and Σ_m for which we easily deduce Σ_{rr} and Σ_θ by solving the linear system

$$\begin{pmatrix}\Sigma_{eq}\\ \Sigma_m\end{pmatrix} = \begin{pmatrix}\sqrt{3} & -\sqrt{3}\\ 1 & 2\end{pmatrix}\begin{pmatrix}\Sigma_{rr}\\ \Sigma_{\theta\theta}\end{pmatrix} \tag{1.22}$$

The solution is

$$\begin{pmatrix}\Sigma_{rr}\\ \Sigma_{\theta\theta}\end{pmatrix} = \frac{1}{3\sqrt{3}}\begin{pmatrix}2 & \sqrt{3}\\ -1 & \sqrt{3}\end{pmatrix}\begin{pmatrix}\Sigma_{eq}\\ \Sigma_m\end{pmatrix} \tag{1.23}$$

or simply

$$\begin{pmatrix}\Sigma_{rr} = \frac{2}{3\sqrt{3}}\Sigma_{eq} + \frac{1}{3}\Sigma_m,\\ \\ \Sigma_{\theta\theta} = \frac{1}{3}\Sigma_m - \frac{1}{3\sqrt{3}}\Sigma_{eq}\end{pmatrix}$$

where the invariants Σ_m and Σ_{eq} are provided by the formula Eq.(1.19) and the solution to the ordinary differential equation Eq.(1.20).

1.5.2 Moment Components Derivation

In this section we solve the components of the moment tensor $\mathbf{M}$. Using the derivations for the invariants Σ_m and Σ_{eq} provided in section § 1.9.1 we have an analytic expression for the function $\mathbf{M}^\star$ (which was defined in the last section). Recall from the constraint equations of the GLPD model Eq.(B.5) in [9]

$$M_{ijj} = 0$$

which leads to the relation

$$M_{rrr} = -2M_{r\theta\theta}.$$

The moment components M_{ijk} are recovered by the use of the flow rule Eq.(1.51) and the velocity field, which we assume can be represented as

$$\mathcal{U} = (f(r), 0, 0)$$

for some function $f(r)$ dependent on the spherical radial coordinate r. This assumption is crude but nonetheless provided one with a mechanical insight into the solution of the problem under consideration. A more generalized velocity field will be considered in future work by the authors of this paper. With this assumption the flow rule Eq.(1.51) will reduce to

$$\nabla D_{mk} = \frac{2}{3}\eta U_k$$

where η is the plastic multiplier and U the velocity field (see [9] for exact details.)

In accordance with Eq.(36) in [9] when $A_1 = 0$ (this assumption is necessary to simplify the solution for the moment components; maintaining a nonzero value for A_1 will not bring any significant difference with respect to the analytic solution of the moment components we shall find in the subsequent results, see for instance [9]) the strain rate components becomes

$$\begin{cases} (\nabla D)_{rrr} & = \dfrac{df}{dr} = \dfrac{\eta}{\Sigma_0^2 b^2} 3A_2 M'_{rrr} \\[2ex] (\nabla D)_{r\theta\theta} & = 0 = \dfrac{\eta}{\Sigma_0^2 b^2} 3A_2 M'_{\theta\theta r} \\[2ex] (\nabla D)_{\theta\theta r} & = \dfrac{1}{r}\dfrac{df}{dr} = \dfrac{\eta}{\Sigma_0^2 b^2} 3A_2 M'_{r\theta\theta} \\[2ex] (\nabla D)_{r\phi\phi} & = \dfrac{1}{r}\dfrac{df}{dr} = \dfrac{\eta}{\Sigma_0^2 b^2} 3A_2 M_{r\theta\theta} \end{cases} \tag{1.25}$$

Let $\kappa = \dfrac{\Sigma_0^2 b^2}{3A_2}$ then the deviatoric parts of the moment $\mathbf{M}$ satisfy

$$M'_{rrr} = \frac{\kappa}{\eta}\frac{df}{dr}, \quad M'_{\theta\theta r} = 0, \quad M'_{r\theta\theta} = \frac{\kappa}{\eta}\frac{1}{r}\frac{df}{dr}, \quad M_{r\theta\theta} = \frac{\kappa}{\eta}\frac{1}{r}\frac{df}{dr}.$$

The value of Q^2 then is computed as follows

$$\begin{aligned} Q^2 &= \frac{3}{2}A_2\left(M'^2_{rrr} + 2M'^2_{\theta\theta r} + 4M'^2_{r\theta\theta}\right) \\ &= \frac{3}{2}A_2\left(\left(\frac{\kappa}{\eta}\frac{df}{dr}\right)^2 + \left(\frac{\kappa}{\eta}\frac{1}{r}\frac{df}{dr}\right)^2\right) \\ &= \frac{3}{2}A_2\left(\frac{df}{dr}\frac{\kappa}{\eta}\right)^2\left(\frac{r^2+1}{r^2}\right) \end{aligned} \tag{1.26}$$

Using the yield criterion

$$\frac{1}{\Sigma^2}\left(\Sigma_{eq}^2 + \frac{Q^2}{b^2}\right) + 2p\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma}\right) - 1 - p^2 = 0.$$

we have

$$Q^2 = b^2\left(\Sigma^2\left(p^2 + 1 - 2p\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma}\right)\right) - \Sigma_{eq}^2\right) \tag{1.27}$$

Therefore the function f then satisfies the ordinary differential equation

$$\left(\frac{df}{dr}\right)^2 = \frac{2}{3A_2}\left(\frac{\eta}{\kappa}\right)^2\left(\frac{r^2}{r^2+1}\right)b^2\left(\Sigma^2\left(p^2 + 1 - 2p\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma}\right)\right) - \Sigma_{eq}^2\right)$$

Using Eq.(37) in [9] ones gets the devatoric part of the stress

$$\Sigma'_{rr} = -\frac{\Sigma_0^2}{3}\frac{1}{\eta}f(r), \quad \Sigma'_{\theta\theta} = \frac{\Sigma_0^2}{3}\frac{1}{\eta}f(r). \tag{1.28}$$

Since

$$\Sigma_{ij} = \Sigma'_{ij} + \Sigma_m\delta_{ij}$$

were Σ'_{ij} is the devatoric part of the stess the mean stress can then be expressed as

$$\Sigma_m = \Sigma_{rr} + \frac{\Sigma_0^2}{3}\frac{1}{\eta}f(r) \tag{1.29}$$

Hence, the plastic multiplier then satisifies

$$\eta = \frac{\Sigma_0^2}{3}\frac{f(r)}{(\Sigma_m - \Sigma_{rr})} \tag{1.30}$$

Using Eq.(1.24) gives us the expression in terms of the invariants

$$\eta = \frac{\Sigma_0^2}{2} \frac{f(r)}{\left(\Sigma_m - \frac{1}{\sqrt{3}\Sigma_{eq}} \right)} \tag{1.31}$$

were Σ_m and Σ_{eq} are provided in the previous section. Substituting Eq.(1.31) into Eq.(1.28) gives one the relation

$$\left(\frac{df}{dr} \right)^2 = f(r)^2 p(r) \tag{1.32}$$

where $p(r)$ is provided by

$$p(r) = h(r)\iota(r)j(r)$$

were

$$\iota(r) = \left(\frac{\Sigma_0^4}{6(\Sigma_m - \frac{1}{\sqrt{3}\Sigma_{eq}})^2 \kappa^2 A_2} \right), \quad j(r) = \left(\frac{r^2}{r^2+1} \right),$$

and

$$h(r) = \left(b^2 \left(\Sigma^2 \left(p^2 + 1 - 2p \cosh \left(\frac{3}{2} \frac{\Sigma_m}{\Sigma} \right) \right) - \Sigma_{eq}^2 \right) \right).$$

The solution for the function f reads

$$f(r) = \exp \left(\int_{r_i}^{r} \sqrt{p(r)} dr \right). \tag{1.33}$$

Next the non zero components of the moment are recovered by the formulas Eq.(1.25)

$$\begin{cases} M'_{rrr} = \frac{\kappa}{\eta} \frac{df}{dr} \\ \\ M'_{\theta\theta r} = M'_{\phi\phi r} = \frac{-\kappa}{\eta} \frac{1}{r} \frac{df}{dr} \\ M'_{r\theta\theta} = M_{r\theta\theta} = \frac{\kappa}{\eta} \frac{1}{r} \frac{df}{dr} \end{cases} \tag{1.35}$$

where η and κ are again given as

$$\begin{cases} \eta = \dfrac{\Sigma_0^2}{2} \dfrac{f(r)}{\left(\Sigma_m - \dfrac{1}{\sqrt{3}\Sigma_{eq}}\right)} \\ \\ \kappa = \dfrac{\Sigma_0^2 b^2}{3A_2} \end{cases}$$

The conditions (1.40) relate the rest of the non-zero components of the moment tensor, M_{rrr} and $M_{r\theta\theta}$ as follows:

$$M_{rrr} + 2M_{rrr} = 0 \tag{1.37}$$

Furthermore, combining Eq. (1.37) and the values of M_{rrr} and $M_{r\theta\theta}$ given by the relations Eq.(1.36) we find

$$M_{mr} = M_{rrr} - M'_{rrr} = 2M_{r\theta\theta} - M'_{rrr} \tag{1.38}$$

where M_{mr} denotes the deviatoric part of the tensor $\mathbf{M}$ over its first two indices. The formula immediately yields $M_{\theta\theta r} = M'_{\theta\theta r}$. The formula Eq.(1.38) immediatly gives

$$M_{r\theta\theta} = M'_{\theta\theta r} \tag{1.39}$$

1.5.3 Discussion

In this section we elaborate on some of the results found in this work. If the characteristic length scale b vanishes then all of the moment components also vanish by Eq.(1.25). This means that any second gradient effects in the GLPD model no longer exist. The yield criterion Eq.(1.52) reduces to the original Gurson model yield criterion and the stress state of the material does not contain any length scale effects (as one can expect). Unlike in the work of [9] where a vanishing porosity is consider, this work was derived while the porosity is kept constant i.e. at the initial time where the porosity has not yet evolved. Such effect is left for future investigations by the authors. However, let us mention that the works by [19, 20] have addressed such problem (the hollow sphere problem with evolving porosity) but the model their analysis is based on does not contain any strain gradient effects.

1.6 Conclusion

In this work we develop the complete solution for the micromorphic hollow sphere model under tension obeying the GLPD constitutive model when the porosity is constant at the initial time and we illustrate the analytic solutions provided previously by Enakousta's work when the porosity is neglected, but some effects of the strain gradient were involved. We express the stress solutions in terms of the invariants of the Cauchy stress tensor. The solution of the nonzero components of the moments due to the strain gradient effects are also provided in this work. The solution produced in this work depends on the characteristic length scale and can be used as benchmark solution to assess micromorphic gradient models; the solution can also be used to test the efficiency of numerical implementation of gradient models into finite element software.

References

1. Ahad, F., Enakoutsa, K., Solanki, K., and Bammann, D. 2013. Nonlocal Modeling in High Velocity Impact failure of Aluminum 6160-T6, *Int. J. Plast.,* **55**, 108-132
2. Andreaus, U., dell'Isola, F., Giorgio, I., Placidi, L., Lekszycki, T., Rizzi, N. L. 2016. Numerical simulations of classical problems in two-dimensional (non) linear second gradient elasticity. International Journal of Engineering Science, 108, 34-50.
3. Bergheau, J-M, Leblond, J-B, Perrin, G. 2013. A New Numerical Implementation of a second-gradient model for plastic porous solids, with an application of ductile rupture tests, *Computer Methods in Applied Mechanics and Engineering,* **268**, 105-125
4. dell'Isola, F., Della Corte, A., Giorgio, I. 2017. Higher-gradient continua: The legacy of Piola, Mindlin, Sedov and Toupin and some future research perspectives. *Mathematics and Mechanics of Solids*, **22**(4), 852-872.
5. Gurson, Arthur L. 1977. Continuum theory of ductile rupture by void nucleation and growth: Part I Yield criteria and flow rules for porous ductile media.*J of Eng. Mater. Technol*, **99**(1), 2-15.
6. Giorgio, I., Andreaus, U., dell'Isola, F., Lekszycki, T. 2017. Viscous second gradient porous materials for bones reconstructed with bio-resorbable grafts. *Extreme Mechanics Letters*, **13**, 141-147.
7. Enakoutsa, K., 2007. Modèle Non-locaux en Rupture Ductile des Métaux. Ph.D thesis, Université Pierre et Marie Curie (Paris VI) (in French)
8. Enakoutsa, K., 2012. Some new Applications of the GLPD Micromorphic Model of Ductile Fracture. *Mathematics and Mechanics of Solids,* **19**(3), 242-259
9. Enakoutsa, K. 2013. Exact results for the problem of a hollow sphere subjected to hydrostatic tension and made of micromorphic plastic porous material. *Mechanics Research Communications* **49**, 1-7.
10. Enakoutsa K., Leblond J.B. and Perrin G. 2007. Numerical implementation and assessment of a phenomenological nonlocal model of ductile rupture, *Comput. Meth. Appl. Mech. Engng.*, **196**, 1946-1957
11. Enakoutsa, K., Solanki, K., Ahad, F., Tjiptowidjojo, Y., and Bammann, D. 2012. Using Damage Delocalization to Model Localization Phenomena in Bammann-Chiesa-Johnson Metals, *J. Eng. Mater. Tech.,* **134**(4)
12. Enakoutsa, K., Solanki, K., Ahad, F., Tjiptowidjojo, Y., and Bammann, D. 2012. Damage smoothing effects in a delocalized rate sensitivity model for metals, *Theoretical and Applied Mechanical Letters,* **2**(5): 5-051005
13. Enakoutsa, K. 2013. Exact results for the problem of a hollow sphere subjected to hydrostatic tension and made of micromorphic plastic porous materials, *Mech. Res. Commun.,* **49**, 1-7
14. Enakoutsa, K. 2013. An Analytic Benchmark Solution to the Problem of a Generalized Plane Strain Hollow Cylinder made of Micromorphic Plastic Porous Metal and Subjected Axisymmetric Loading Conditions, *Math. Mech. Solids,* **9**, 1013-1025
15. Enakoutsa K., and Leblond J.B., 2009. Numerical Implementation and Assessment of the GLPD Micromorphic Model of Ductile Rupture. *European Journal of Mechanics A/Solids*, **28**, 445-460
16. Gologanu M., Leblond J.B., Perrin G. and Devaux J. 1997. Recent extensions of Gurson's model for porous ductile metals, in: *Continuum Micromechanics*, CISM Courses and Lectures 377, P. Suquet ed., Springer, pp. 61-130
17. Gurson A.L. 1977. Continuum theory of ductile rupture by void nucleation and growth: Part I - yield criteria and flow rules for porous ductile media, *ASME J. Engng. Mater. Technol.*, **99**, 2-15
18. Leblond, J.B., Perrin, G., and Devaux, J. 1994. Bifurcation Effects in Ductile Metals with Nonlocal Damage, *ASME J. Applied . Mech.*, **6**1, 236-242.

19. Perrin, G, and Leblond, JB 2000. Accelerated void growth in porous ductile solids containing two populations of cavities. *Int. J. Plast.,* **16**, 91v120
20. Perrin, G, and Leblond, J-B, 1990. Analytical study of a hollow sphere made of plastic porous material and subjected to hydrostatic tension Application to some problems in ductile fracture of metals, *Int. J. Plast.,* **6**:677699
21. Pijaudier-Cabot, G. and Bazant, Z.P., 1987. Nonlocal Damage Theory, *ASCE J. Engrg. Mech.*, **1**13, 1512-1533
22. Placidi, L., Barchiesi, E., Misra, A., Timofeev, D. 2021. Micromechanics-based elasto-plasticity damage energy formulation for strain gradient solids with granular microstructure. Continuum Mechanics and Thermodynamics, **33**, 2213-2241
23. Nahshon, K., Hutchinson, J.W., 2008. Modification of the Gurson model for shear failure, *Eur. J. Mech. A/Solids,* **27**, 1-17.
24. Rice, J. R. and Tracey, D. M. 1969. On the Ductile Enlargement of Voids in Triaxial Growth Holes. *J. Mech. Phys. Solids,* **17**, 210-217
25. Tvergaard V. and Needleman A. 1997. Nonlocal effects on localization in a void-sheet, *Int. J. Solids Structures,* **34**, 2221-2238
26. Tvergaard V. and Needleman A. 1995. Effects of nonlocal damage in porous plastic solids, *Int. J. Solids Structures,* **32**, 1063-1077 s
27. Reiher, J. C., Giorgio, I., Bertram, A. 2017. Finite-element analysis of polyhedra under point and line forces in second-strain gradient elasticity. Journal of Engineering Mechanics, **143**(2), 04016112.
28. Scerrato, D., Bersani, A. M., Giorgio, I. 2021. Bio-Inspired Design of a Porous Resorbable Scaffold for Bone Reconstruction: A Preliminary Study. *Biomimetics*, **6**(1), 18.
29. Tvergaard V. 1981. Influence of voids on shear band instabilities under plane strain conditions, *Int. J. Fracture*, **17**, 389-407.
30. Tvergaard V. and Needleman A. 1984. Analysis of cup-cone fracture in a round tensile bar, *Acta Metall.*,**32**, 157-169.

1.7 Appendix (a)

1.7.1 Generalities

In the GLPD model, internal forces are represented through some ordinary second-rank symmetric Cauchy stress tensor $\boldsymbol{\Sigma}$ plus some additional third-rank "moment tensor" $\mathbf{M}$ symmetric in its first two indices only[2]. The components of $\mathbf{M}$ are related through the three conditions

$$M_{ijj} = 0. \tag{1.40}$$

(These conditions may be compared to the condition of plane stress in the theory of thin plates or shells).

The virtual power of internal forces is given by the expression

$$\mathcal{P}^{(i)} \equiv -\int_{\Omega} (\boldsymbol{\Sigma} : \mathbf{D} + \mathbf{M} \,\vdots\, \nabla\mathbf{D})\, d\Omega \tag{1.41}$$

where Ω denotes the domain considered, $\mathbf{D} \equiv \frac{1}{2}\left[\nabla\mathbf{V} + (\nabla\mathbf{V})^T\right]$ ($\mathbf{V}$: material velocity) the Eulerian strain rate, $\nabla\mathbf{D}$ its gradient, $\boldsymbol{\Sigma} : \mathbf{D}$ the double inner product $\Sigma_{ij}D_{ij}$ and $\mathbf{M} \,\vdots\, \nabla\mathbf{D}$ the triple inner product $M_{ijk}D_{ij,k}$.

The virtual power of external forces is given by

$$\mathcal{P}^{(e)} \equiv \int_{d\Omega} \mathbf{T}.\mathbf{V}\, dS \tag{1.42}$$

where $\mathbf{T}$ represents some surface traction[3].

The hypothesis of additivity of elastic and plastic strain rates reads

$$\begin{cases} \mathbf{D} \equiv \mathbf{D}^e + \mathbf{D}^p \\ \nabla\mathbf{D} \equiv (\nabla\mathbf{D})^e + (\nabla\mathbf{D})^p. \end{cases} \tag{1.43}$$

The elastic and plastic parts $(\nabla\mathbf{D})^e$, $(\nabla\mathbf{D})^p$ of the gradient of the strain rate here do *not* coincide in general with the gradients $\nabla(\mathbf{D}^e)$, $\nabla(\mathbf{D}^p)$ of the elastic and plastic parts of the strain rate.

---

[2] The component M_{ijk} is noted $M_{k|ij}$ in [16]'s original paper. The present notation leads to more natural-looking expressions.

[3] The general equilibrium equations and boundary conditions corresponding to the expressions (1.41) and (1.42) of the virtual powers of internal and external forces need not be given since they are not necessary for the numerical implementation.

1.7.2 Hypoelasticity law

The elastic parts of the strain rate and its gradient are related to the rates of the stress and moment tensors through the following hypoelasticity law:

$$\begin{cases} \dfrac{d\Sigma_{ij}}{dt} = \lambda\,\delta_{ij}D^e_{kk} + 2\mu D^e_{ij} \\ \dfrac{dM_{ijk}}{dt} = \dfrac{b^2}{5}\Big[\lambda\,\delta_{ij}(\nabla D)^e_{hhk} + 2\mu(\nabla D)^e_{ijk} \\ \qquad\qquad -2\lambda\,\delta_{ij}U^e_k - 2\mu\left(\delta_{ik}U^e_j + \delta_{jk}U^e_i\right)\Big]. \end{cases} \tag{1.44}$$

In these expressions λ and μ denote the Lamé coefficients and b the mean half-spacing between neighboring voids. (In the homogenization procedure, b is the radius of the spherical elementary cell considered). Also, $\frac{d\Sigma_{ij}}{dt}$ and $\frac{dM_{ijk}}{dt}$ are the Jaumann (objective) time-derivatives of Σ_{ij} and M_{ijk}, given by

$$\begin{cases} \dfrac{d\Sigma_{ij}}{dt} \equiv \dot{\Sigma}_{ij} + \Omega_{ki}\Sigma_{kj} + \Omega_{kj}\Sigma_{ik} \\ \dfrac{dM_{ijk}}{dt} \equiv \dot{M}_{ijk} + \Omega_{hi}M_{hjk} + \Omega_{hj}M_{ihk} + \Omega_{hk}M_{ijh} \end{cases} \tag{1.45}$$

where $\boldsymbol{\Omega} \equiv \frac{1}{2}\left[\nabla\mathbf{V} - (\nabla\mathbf{V})^T\right]$ is the antisymmetric part of the velocity gradient. Finally $\mathbf{U}^e$ is a vector the value of which is fixed by equations (1.40) (written in rate form, $\frac{DM_{ijj}}{Dt} = 0$):

$$U^e_i = \frac{\lambda(\nabla D)^e_{hhi} + 2\mu(\nabla D)^e_{ihh}}{2\lambda + 8\mu}. \tag{1.46}$$

(This vector may be compared to the through-the-thickness component of the elastic strain rate in the theory of thin plates or shells, the value of which is fixed by the condition of plane stress).

1.7.3 Yield criterion

The plastic behavior is governed by the following Gurson-like criterion:

$$\frac{1}{\Sigma^2}\left(\Sigma^2_{eq} + \frac{Q^2}{b^2}\right) + 2p\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma}\right) - 1 - p^2 \le 0. \tag{1.47}$$

In this expression:

- $\Sigma_{eq} \equiv \left(\frac{3}{2}\boldsymbol{\Sigma}' : \boldsymbol{\Sigma}'\right)^{1/2}$ ($\boldsymbol{\Sigma}'$: deviator of $\boldsymbol{\Sigma}$) is the von Mises equivalent stress.
- $\Sigma_m \equiv \frac{1}{3}\operatorname{tr}\boldsymbol{\Sigma}$ is the mean stress.

- Σ represents a kind of average value of the yield stress in the heterogeneous metallic matrix, the evolution equation of which is given below.
- p is a parameter connected to the porosity (void volume fraction) f through the relation:

$$p \equiv qf^*, f^* \equiv \begin{cases} f & \text{if } f \leq f_c \\ f_c + \delta(f - f_c) & \text{if } f > f_c \end{cases} \tag{1.48}$$

  where q is *Tvergaard's parameter*, f_c the *critical porosity at the onset of coalescence of voids*, and δ (> 1) a factor describing the accelerated degradation of the material during coalescence [29],[30].
- Q^2 is a quadratic form of the components of the moment tensor given by

$$Q^2 \equiv A_1\mathcal{M}_1 + A_2\mathcal{M}_2 \quad , \quad \begin{cases} A_1 = 0.194 \\ A_2 = 6.108 \end{cases} \tag{1.49}$$

  where $\mathcal{M}_1$ and $\mathcal{M}_2$ are the quadratic invariants of $\mathbf{M}$ defined by:

$$\begin{cases} \mathcal{M}_1 \equiv M_{mi}M_{mi} \\ \mathcal{M}_2 \equiv \frac{3}{2}M'_{ijk}M'_{ijk}, \end{cases} \tag{1.50}$$

  $M_{mi} \equiv \frac{1}{3}M_{hhi}$ and $\mathbf{M}'$ denoting the mean and deviatoric parts of $\mathbf{M}$, taken over its first two indices.
- Again, b is the mean half-spacing between neighboring voids.

1.7.4 Flow rule

The plastic parts of the strain rate and its gradient are given by the flow rule associated to the criterion (1.47) through normality:

$$\begin{Bmatrix} D^p_{ij} & = \eta \dfrac{d\Phi}{d\Sigma_{ij}}(\boldsymbol{\Sigma}, \mathbf{M}, \Sigma, f) \\ (\nabla D)^p_{ijk} & = \eta \dfrac{d\Phi}{dM_{ijk}}(\boldsymbol{\Sigma}, \mathbf{M}, \Sigma, f) + \delta_{ik}U^p_j \\ & \quad +\delta_{jk}U^p_i \end{Bmatrix} \tag{1.51}$$

where

$$\eta = \begin{Bmatrix} = 0 & \text{if} & \Phi(\boldsymbol{\Sigma}, \mathbf{M}, \Sigma, f) < 0 \\ \geq 0 & \text{if} & \Phi(\boldsymbol{\Sigma}, \mathbf{M}, \Sigma, f) = 0 \end{Bmatrix}$$

The term $\delta_{ik}U^p_j + \delta_{jk}U^p_i$ in equation (1.51) represents a rigid-body motion of the elementary cell, which is left unspecified by the flow rule but fixed in practice by conditions (1.40). (The vector $\mathbf{U}^p$ may be compared to the through-the-thickness component of the plastic strain rate in the theory of thin plates or shells, the value of which is fixed by the condition of plane stress).

The values of the derivatives of the yield function $\Phi(\boldsymbol{\Sigma}, \mathbf{M}, \Sigma, f)$ in equations (1.51) are easily calculated to be

$$\left\{ \begin{aligned} \frac{d\Phi}{d\Sigma_{ij}}(\boldsymbol{\Sigma}, \mathbf{M}, \Sigma, f) &= 3\frac{\Sigma'_{ij}}{\Sigma^2} + \frac{p}{\Sigma}\delta_{ij}\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma}\right) \\ \frac{d\Phi}{dM_{ijk}}(\boldsymbol{\Sigma}, \mathbf{M}, \Sigma, f) &= \frac{1}{\Sigma^2 b^2}\left(\frac{2}{3}A_1\delta_{ij}M_{mk} + 3A_2 M'_{ijk}\right) \end{aligned} \right\} \tag{1.52}$$

1.7.5 Evolution of Internal Parameters

The evolution of the porosity is governed by the classical equation resulting from approximate incompressibility of the metallic matrix:

$$\dot{f} = (1-f)\,\mathrm{tr}\,\mathbf{D}^p. \tag{1.53}$$

The parameter Σ is given by

$$\Sigma \equiv \Sigma(E) \tag{1.54}$$

where $\Sigma(\epsilon)$ is the function which provides the yield stress of the matrix material in terms of the local equivalent cumulated plastic strain ϵ, and E represents some average value of this equivalent strain in the heterogeneous matrix. The evolution of E is governed by the following equation:

$$(1-f)\Sigma\dot{E} = \boldsymbol{\Sigma} : \mathbf{D}^p + \mathbf{M} \vdots (\nabla\mathbf{D})^p. \tag{1.55}$$

1.8 Apendix (b)

In this section we present the complicate analytic derivations of the integrals for the hollow sphere problem while neglecting the porosity. Although, the formulation is long and tedious we found some need to put it into a single equation. This equation reads:

$$\left\{ \begin{aligned} &f(r) = \frac{2A\Sigma_0^2}{\eta r^3} + \frac{2(\eta''\eta^2 - 2\eta'^2\eta)}{\eta^4}\frac{A\Sigma_0^2 b^2}{A_{II} r^4} - \frac{28\eta'}{\eta^2}\frac{A\Sigma_0^2 b^2}{A_{II} r^5} \\ &-\left(\frac{72}{\eta} + \frac{2\eta'}{\eta^2}\right)\frac{A\Sigma_0^2 b^2}{A_{II} r^6} - \frac{8A\Sigma_0^2 b^2}{\eta A_{II} r^7} \end{aligned} \right. \tag{1.56}$$

To obtain the result we integrated this formulation term by term. Evaluating the integral of the first term in the expression for f we found

$$\begin{aligned}\int_{r_i}^{r} \frac{2A\Sigma_0^2}{\eta(t)t^3}dt &= 2\Sigma_0 \int_{r_i}^{r} \frac{dt}{\sqrt{1+\frac{15b^2}{A_2t^2}}} \\ &= 2\Sigma_0\left(\sqrt{\frac{A_2r^2+15b^2}{A_2}} - \sqrt{\frac{A_2r_{\mathrm{i}}^2+15b^2}{A_2}}\right) \qquad (1.57)\end{aligned}$$

The second term in the expression for f is broken into two smaller integrals. The result reads

$$\begin{aligned}\int_{r_i}^{r} \frac{2A\Sigma_0^2b^2}{A_2}\frac{\eta(t)''\eta(t)-\eta(t)'}{\eta(t)^2t^4}dt &= \frac{2A\Sigma_0^2b^2}{A_2}\left[\int_{r_i}^{r}\frac{\eta(t)''}{\eta(t)t^4}dt - \int_{r_i}^{r}\frac{\eta(t)'}{\eta(t)t^4}dt\right] \\ &= \frac{2A\Sigma_0^2b^2}{A_2^3}\left[\int_{r_i}^{r}\frac{12A_2^2t^4+495A_2b^2t^2+4500b^4}{t^{10}(1+\frac{15b^2}{A_2t^2})^2}dt\right. \\ &\quad \left.+\frac{1}{A_2}\int_{r_i}^{r}\frac{60b^2+3A_2t^2}{t^7(1+\frac{15b^2}{A_2t^2})}dt\right] \qquad (1.58)\end{aligned}$$

Evaluating these two integrals we obtain

$$\begin{aligned}\int_{r_i}^{r}\frac{12A_2^2t^4+495A_2b^2t^2+4500b^4}{t^{10}(1+\frac{15b^2}{A_2t^2})^2}dt &= \left[\frac{A_2^5r}{450A_2b^4r^2+6750b^6} - \frac{11A_2^{\frac{9}{2}}\arctan\left(\frac{\sqrt{A_2}r}{\sqrt{15}b}\right)}{2\cdot 15^{\frac{5}{2}}b^5}\right. \\ &\quad -\frac{2A_2^4}{75b^4r}+\frac{7A_2^3}{45b^2r^3}-\frac{4A_2^2}{r^5}+\frac{11a_2^{\frac{9}{2}}\arctan\left(\frac{\sqrt{A_2}r_i}{\sqrt{15}b}\right)}{2\cdot 15^{\frac{5}{2}}b^5} \\ &\quad \left.-\frac{A_2^5r_i}{450A_2b^4r_i^2+6750b^6}+\frac{2A_2^4}{75b^4r_i}-\frac{7A_2^3}{45b^2r_i^3}+\frac{4A_2^2}{r_i^5}\right] \qquad (1.59)\end{aligned}$$

and

$$\begin{aligned}\int_{r_i}^{r}\frac{60b^2+3A_2t^2}{t^7(1+\frac{15b^2}{A_2t^2})}dt &= \left[\frac{A_2^3\ln\left(\left|A_2r_i^2+15b^2\right|\right)-A_2^3\ln\left(\left|A_2r^2+15b^2\right|\right)}{450b^4}\right. \\ &\quad \left.+\frac{A_2^3\ln(r)-A_2^3\ln(r_i)}{225b^4}-\frac{A_2^2}{20b^2r_i^2}+\frac{A_2}{r_i^4}+\frac{A_2^2}{30b^2r^2}-\frac{A_2}{r^4}\right] \qquad (1.60)\end{aligned}$$

The integration of the third term for the function f is broken into two smaller integrals. The result reads

$$\int_{r_i}^{r} \left(\frac{A\Sigma_0^2 b^2}{A_2}\right)\frac{20\eta(t)' + 8\eta(t)'\eta(t)}{\eta(t)^2 t^2}dt = \frac{A\Sigma_0^2 b^2}{A_2}\left[\int_{r_i}^{r}\frac{20\eta(t)'}{\eta(t)^2 t^2}dt + \int_{r_i}^{r}\frac{8\eta(t)'}{\eta(t)t^2}dt\right]$$
$$= -\left[\frac{20\Sigma_0 b^2}{A_2^2}\int_{r_i}^{r}\frac{60b^2 + 3A_2t^2}{t^2(1+\frac{15b^2}{A_2t^2})^{\frac{3}{2}}}dt + \frac{8A\Sigma_0^2 b^2}{A_2}\int_{r_i}^{r}\frac{60b^2+3A_2t^2}{A_2t^5+15bt^3}dt\right] \quad (1.61)$$

Now evaluating these two less complicated integrals we obtain

$$\int_{r_i}^{r}\frac{60b^2+3A_2t^2}{t^2(1+\frac{15b^2}{A_2t^2})^{\frac{3}{2}}}dt = \left[\frac{3^{\frac{3}{2}}\cdot\sqrt{5}\left(A_2r^2+10b^2\right)|r|}{\frac{r\sqrt{15}\sqrt{A_2r^2+16b^2}}{A_2}} - \frac{3^{\frac{3}{2}}\cdot\sqrt{5}\left(A_2r_i^2+10b^2\right)|r_i|}{\frac{r_i\sqrt{15}\sqrt{A_2r_i^2+16b^2}}{A_2}}\right] \quad (1.62)$$

and

$$\int_{r_i}^{r}\frac{60b^2+3A_2t^2}{A_2t^5+15bt^3}dt = \left[\frac{A_2^3\ln(|A_2r^2+15b^2||) - A^3\ln(|ar_i^2+15b^2|)}{6750b^6} + \frac{A_2^3\ln(r_i) - A_2^3\ln(r)}{3375b^6} + \frac{A_2^2}{450b^4r_i^2} - \frac{A_2}{60b^2r_i^4} + \frac{2}{3r^6} - \frac{A_2^2}{450b^4r^4} + \frac{A_2}{60b^2r^4} - \frac{2}{3r^6}\right] \quad (1.63)$$

The integration of the fourth term for the function f is broken into two smaller integrals. The result reads

$$\int_{r_i}^{r}\frac{(72+2\eta'(t))}{\eta(t)}\left(\frac{A\Sigma_0^2b^2}{A_2t^6}\right)dt = \frac{A\Sigma_0 b^2}{A_2}\left[\int_{r_i}^{r}\frac{72}{\eta(t)t^6}dt + \int_{r_i}^{r}\frac{2\eta(t)'}{\eta(t)t^6}dt\right]$$
$$= \frac{72b^2}{A_2}\left[\int_{r_i}^{r}\frac{dt}{t^3\sqrt{1+\frac{15b^2}{A_2t^2}}} - \frac{2A\Sigma_0b^2}{A_2^2}\int_{r_i}^{r}\frac{60b^2+3A_2t^2}{t^9(1+\frac{15b^2}{A_2t^2})}\right] \quad (1.64)$$

Integrating these two integrals we obtain

$$\int_{r_i}^{r}\frac{dt}{t^3\sqrt{1+\frac{15b^2}{A_2t^2}}} = \frac{A_2\left[r\sqrt{\frac{A_2r_{\mathrm{i}}^2+15b^2}{A_2}} - \sqrt{\frac{A_2r^2+15b^2}{A_2}}r_{\mathrm{i}}\right]}{15b^2rr_{\mathrm{i}}} \quad (1.65)$$

and

$$\int_{r_i}^{r} \frac{60b^2+3A_2t^2}{t^9(1+\frac{15b^2}{A_2t^2})}dt = \Bigg[\frac{A_2^4 \ln\left(\left|A_2r^2+15b^2\right|\right)}{6750b^6} \tag{1.66}$$

$$-\frac{A_2^4 \ln\left(\left|A_2r_i^2+15b^2\right|\right)}{6750b^6} + \frac{A_2^4 \ln(r_i) - A_2^4 \ln(r)}{3375b^6}$$

$$+\frac{A_2^3}{450b^4r_i^2} - \frac{A_2^2}{60b^2r_i^4} + \frac{2A_2}{3r_i^6} - \frac{A_2^3}{450b^4r^2} + \frac{A_2^2}{60b^2r^4} - \frac{2A_2}{3r^6}\Bigg] \tag{1.67}$$

Now integrating the last term in the expression for f we get

$$\int_{r_i}^{r} \frac{2A\Sigma_0^2b^2}{\eta(t)t^2}dt = 2\Sigma_0b^2 \int_{r_i}^{r} \frac{t}{\sqrt{1+\frac{15b^2}{A_2t^2}}}dt$$

$$= \left(\frac{2\sqrt{a_2}r_\mathrm{i}\sqrt{a_2r_\mathrm{i}^2+15b^2}}{4A_2} - \frac{15b^2\ln\left(\frac{\left|\sqrt{a_2}\sqrt{a_2r_\mathrm{i}^2+15b^2}-a_2r_\mathrm{i}\right|}{a_2r_\mathrm{i}}\right)}{4a^2}\right.$$

$$-\frac{15b^2\ln\left(\frac{\sqrt{a_2}\sqrt{a_2r_\mathrm{i}^2+15b^2}+a_2r_\mathrm{i}}{a_2r_\mathrm{i}}\right)}{4A_2} - \frac{15b^2\ln\left(\frac{\left|\sqrt{a_2}\sqrt{a_2r^2+15b^2}-a_2r\right|}{a_2r}\right)}{4A_2}$$

$$\left.+\frac{15b^2\ln\left(\frac{\sqrt{a_2}\sqrt{a_2r^2+15b^2}+a_2r}{a_2r}\right)}{4A_2} - \frac{-2\sqrt{a_2}r\sqrt{a_2r^2+15b^2}}{4a_2}\right) \tag{1.68}$$

1.9 Appendix (c)

1.9.1 Stress Equations

In this section we present the mathematical detail that was missing from the solution to the hollow sphere problem when the porosity is present. The Von Mises stress Σ_{eq}^2 reduces

$$\begin{aligned}\Sigma_{eq}^2 &= \frac{3}{2}(\Sigma'_{ij} : \Sigma'_{ij}) \\ &= \frac{3}{2}\left((\Sigma_{rr}-\Sigma_{\theta\theta})^2 + (\Sigma_{\theta\theta}-\Sigma_{\phi\phi})^2 + (\Sigma_{\phi\phi}-\Sigma_{rr})^2\right) \\ &= 3\left(\Sigma_{rr}-\Sigma_{\theta\theta}\right)^2\end{aligned} \tag{1.69}$$

By the last equality in Eq.(1.69) we find

$$\Sigma_{eq} = \sqrt{3}\left(\Sigma_{rr}-\Sigma_{\theta\theta}\right) \tag{1.70}$$

Solving for Σ_{rr} in Eq.(1.70) we have $\Sigma_{rr} = \frac{1}{\sqrt{3}}\Sigma_{eq} + \Sigma_{\theta\theta}$. Taking the derivative $\frac{d}{dr}$ we see that

$$\frac{d}{dr}\Sigma_{rr} = \frac{1}{\sqrt{3}}\frac{d}{dr}\Sigma_{eq} + \frac{d}{dr}\Sigma_{\theta\theta} \tag{1.71}$$

Now since $\frac{1}{\sqrt{3}}\Sigma_{eq} = \Sigma_{rr} - \Sigma_{\theta\theta}$ by adding $3\Sigma_{\theta\theta}$ to both sides we get $\frac{1}{\sqrt{3}}\Sigma_{eq} + 3\Sigma_{\theta\theta} = \Sigma_{rr} + 2\Sigma_{\theta\theta}$. Multiplying throughout by $\frac{1}{3}$ yields $\frac{1}{3\sqrt{3}}\Sigma_{eq} + \Sigma_{\theta\theta} = \frac{1}{3}\left(\Sigma_{rr} + 2\Sigma_{\theta\theta}\right)$. Since, $\Sigma_m = \frac{1}{3}\left(\Sigma_{rr} + 2\Sigma_{\theta\theta}\right)$ it then follows that $\frac{1}{3\sqrt{3}}\Sigma_{eq} + \Sigma_{\theta\theta} = \Sigma_m$. Rearranging the last equality gives us $\Sigma_{\theta\theta} = \Sigma_m - \frac{1}{3\sqrt{3}}\Sigma_{eq}$. Taking the derivative of both sides we find that

$$\frac{d}{dr}\Sigma_{\theta\theta} = \frac{d}{dr}\Sigma_m - \frac{1}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq} \tag{1.72}$$

Thus we can compute the value of $\frac{d\Sigma_{rr}}{dr} + \frac{2}{r}(\Sigma_{rr} - \Sigma_{\theta\theta})$ as follows

$$\begin{aligned}
\frac{d\Sigma_{rr}}{dr} + \frac{2}{r}(\Sigma_{rr} - \Sigma_{\theta\theta}) &= \frac{d\Sigma_{rr}}{dr} + \frac{2}{r\sqrt{3}}\Sigma_{eq} \\
&= \left(\frac{1}{\sqrt{3}}\frac{d}{dr}\Sigma_{eq} + \frac{d}{dr}\Sigma_{\theta\theta}\right) + \frac{2r}{\sqrt{3}}\Sigma_{eq} \\
&= \frac{2}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq} + \frac{2}{r\sqrt{3}}\Sigma_{eq} + \frac{d}{dr}\Sigma_m
\end{aligned} \tag{1.73}$$

1.9.2 Von Mises and mean stress formula

In this section we work out the mathematical detail that was left off in section § 1.5. First let $\mathbf{M}^\star$ be given by the rule

$$\mathbf{M}^\star = \frac{d^2 M_{rrr}}{dr^2} + \frac{6}{r}\frac{dM_{rrr}}{dr} + \frac{6}{r^2}M_{rrr}\frac{4}{r} - \frac{dM_{r\theta\theta}}{dr} - \frac{8}{r^2}M_{r\theta\theta} \tag{1.74}$$

According to section § 1.5 we may rewrite Eq.(1.74) as follows

$$\mathbf{M}^\star = \frac{2}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq} + \frac{2}{r\sqrt{3}}\Sigma_{eq} + \frac{d}{dr}\Sigma_m \tag{1.75}$$

Now we calculate Σ_{eq} and $\frac{d}{dr}\Sigma_{eq}$ using the yield criteria. The result reads

$$\Sigma_{eq} = \sqrt{(p\Sigma_0)^2 + \Sigma_0 - 2p\Sigma_0^2\cos\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right) - \left(\frac{Q}{b}\right)^2} \tag{1.76}$$

and

$$\frac{d}{dr}\Sigma_{eq} = \frac{2p\Sigma_0^2 \sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m + \frac{1}{b^2}\frac{d}{dr}Q^2}{2\sqrt{\left(p\Sigma_0\right)^2 + \Sigma_0 - 2p\Sigma_0^2\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right) - \left(\frac{Q}{b}\right)^2}} \tag{1.77}$$

Substituting Eq.(1.76) and Eq.(1.77) into Eq.(1.75) gives us the result

$$\begin{aligned}\mathbf{M}^\star = \frac{1}{3\sqrt{3}}&\left(\frac{2p\Sigma_0^2 \sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m + \frac{1}{b^2}\frac{d}{dr}Q^2}{\sqrt{(p\Sigma_0)^2 + \Sigma_0 - 2p\Sigma_0^2\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right) - \left(\frac{Q}{b}\right)^2}}\right)\\ &+\frac{2}{r}\left(\sqrt{(p\Sigma_0)^2 + \Sigma_0 - 2p\Sigma_0^2\cos\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right) - \left(\frac{Q}{b}\right)^2}\right) + \frac{d}{dr}\Sigma_m\end{aligned} \tag{1.78}$$

By subtracting Eq.(1.78) from Eq.(1.75) we find

$$\begin{aligned}\frac{2}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq} + \frac{2}{r\sqrt{3}}\Sigma_{eq} = \frac{1}{3\sqrt{3}}&\left(\frac{2p\Sigma_0^2 \sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m + \frac{1}{b^2}\frac{d}{dr}Q^2}{\sqrt{(p\Sigma_0)^2 + \Sigma_0 - 2p\Sigma_0^2\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right) - \left(\frac{Q}{b}\right)^2}}\right)\\ &+\frac{2}{r}\left(\sqrt{(p\Sigma_0)^2 + \Sigma_0 - 2p\Sigma_0^2\cos\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right) - \left(\frac{Q}{b}\right)^2}\right)\end{aligned} \tag{1.79}$$

Moving over terms and then squaring both sides of Eq.(1.79) we have

$$\begin{aligned}&\left(\frac{2}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq} + \frac{2}{r\sqrt{3}}\Sigma_{eq} - \frac{1}{3\sqrt{3}}\left(\frac{2p\Sigma_0^2 \sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m + \frac{1}{b^2}\frac{d}{dr}Q^2}{\sqrt{(p\Sigma_0)^2 + \Sigma_0 - 2p\Sigma_0^2\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right) - \left(\frac{Q}{b}\right)^2}}\right)\right)^2\\ &\qquad = \left(\Sigma_0^2 + \Sigma_0 - 2p\Sigma_0^2\cos\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right) - \left(\frac{Q}{b}\right)^2\right).\end{aligned} \tag{1.80}$$

Finally dividing Eq.(1.80) throughout by Σ_0^2 and after adding the value $p^2 + \left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2$ to both sides of the equation we find

$$\begin{aligned}&\frac{1}{\Sigma_0^2}\left(\frac{2}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq} + \frac{2}{r\sqrt{3}}\Sigma_{eq} - \frac{1}{3\sqrt{3}}\left(\frac{2p\Sigma_0^2 \sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m + \frac{1}{b^2}\frac{d}{dr}Q^2}{\sqrt{(p\Sigma_0)^2 + \Sigma_0 - 2p\Sigma_0^2\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right) - \left(\frac{Q}{b}\right)^2}}\right)\right)^2\\ &\qquad = \frac{1}{\Sigma_0} - \left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2\end{aligned} \tag{1.81}$$

taking the square root of both sides of Eq.(1.81) and multiplying by Σ_0 gives

$$\frac{2}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq}+\frac{2}{r\sqrt{3}}\Sigma_{eq}-\frac{1}{3\sqrt{3}}\left(\frac{2p\Sigma_0^2\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m+\frac{1}{b^2}\frac{d}{dr}Q^2}{\sqrt{(p\Sigma_0)^2+\Sigma_0-2p\Sigma_0^2\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)-\left(\frac{Q}{b}\right)^2}}\right)$$
$$=\Sigma_0\sqrt{\frac{1}{\Sigma_0}-\left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2} \tag{1.82}$$

Now we do the same procedure to get rid of any terms involving the quadratic term Q. By moving over terms in Eq.(1.82) it follows that

$$\frac{2}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq}+\frac{2}{r\sqrt{3}}\Sigma_{eq}-\Sigma_0\sqrt{\frac{1}{\Sigma_0}-\left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2}$$
$$=\frac{1}{3\sqrt{3}}\left(\frac{2p\Sigma_0^2\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m+\frac{1}{b^2}\frac{d}{dr}Q^2}{\sqrt{(p\Sigma_0)^2+\Sigma_0-2p\Sigma_0^2\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)-\left(\frac{Q}{b}\right)^2}}\right) \tag{1.83}$$

Dividing both sided of Eq.(1.83) by

$$\frac{1}{3\sqrt{3}}\left(2p\Sigma_0^2\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m+\frac{1}{b^2}\frac{d}{dr}Q^2\right)$$

we obtain

$$\frac{\frac{1}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq}+\frac{2}{r\sqrt{3}}\Sigma_{eq}-\Sigma_0\sqrt{\frac{1}{\Sigma_0}-\left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2}}{\frac{1}{3\sqrt{3}}\left(2p\Sigma_0^2\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m+\frac{1}{b^2}\frac{d}{dr}Q^2\right)}$$
$$=\frac{1}{\sqrt{(p\Sigma_0)^2+\Sigma_0-2p\Sigma_0^2\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)-\left(\frac{Q}{b}\right)^2}} \tag{1.84}$$

reciprocating both sides of Eq.(1.84) and then after squaring both sides gives the relation

$$\left(\frac{\frac{1}{3\sqrt{3}}\left(2p\Sigma_0^2\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m+\frac{1}{b^2}\frac{d}{dr}Q^2\right)}{\frac{1}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq}+\frac{2}{r\sqrt{3}}\Sigma_{eq}-\Sigma_0\sqrt{\frac{1}{\Sigma_0}-\left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2}}\right)^2$$
$$=(p\Sigma_0)^2+\Sigma_0-2p\Sigma_0^2\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)-\left(\frac{Q}{b}\right)^2 \tag{1.85}$$

Again dividing Eq.(1.85) throughout by Σ_0^2 and after adding the value $p^2 + \left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2$ to both sides gives us the relation

$$\frac{1}{\Sigma_0^2}\left(\frac{\frac{1}{3\sqrt{3}}\left(2p\Sigma_0^2\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m + \frac{1}{b^2}\frac{d}{dr}Q^2\right)}{\frac{1}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq} + \frac{2}{r\sqrt{3}}\Sigma_{eq} - \Sigma_0\sqrt{\frac{1}{\Sigma_0} - \left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2}}\right)^2 = \frac{1}{\Sigma_0} - \left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2 \tag{1.86}$$

Now assume $\Sigma_{eq} \geq \sqrt{\Sigma_0}$ so that $\frac{1}{\Sigma_0} - \left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2 \geq 0$. In this case solving for $\frac{1}{b^2}\frac{d}{dr}Q^2$ in Eq.(1.86) gives us

$$\begin{aligned}\frac{1}{b^2}\frac{d}{dr}Q^2 = \frac{3\sqrt{3}}{1}&\left(\frac{1}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq} + \frac{2}{r\sqrt{3}}\Sigma_{eq} - \Sigma_0\sqrt{\frac{1}{\Sigma_0} - \left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2}\right)\left(\Sigma_0\sqrt{\frac{1}{\Sigma_0} - \left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2}\right)\\ &-\left(2p\Sigma_0^2\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m\right)\end{aligned} \tag{1.87}$$

But also by the yield criteria 1.47 we know that

$$-\frac{1}{b^2}\frac{d}{dr}Q^2 = 2p\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\left(\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m\right) - 2\Sigma_{eq}\frac{d}{dr}\Sigma_{eq} \tag{1.88}$$

Finally by adding Eq.(1.88) and Eq.(1.87) we have

$$\begin{aligned}&\frac{3\sqrt{3}}{1}\left(\frac{1}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq} + \frac{2}{r\sqrt{3}}\Sigma_{eq} - \Sigma_0\sqrt{\frac{1}{\Sigma_0} - \left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2}\right)\left(\Sigma_0\sqrt{\frac{1}{\Sigma_0} - \left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2}\right)\\ &= 2\Sigma_{eq}\frac{d}{dr}\Sigma_{eq} + \left(2p\Sigma_0^2\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m\right) - 2p\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\end{aligned} \tag{1.89}$$

Since $\Sigma_0\sqrt{\frac{1}{\Sigma_0} - \left(\frac{\Sigma_{eq}}{\Sigma_0}\right)^2} = \sqrt{\Sigma_0 - \Sigma_{eq}^2}$ Eq.(1.89) becomes

$$\begin{aligned}&\frac{3\sqrt{3}}{1}\left(\frac{1}{3\sqrt{3}}\frac{d}{dr}\Sigma_{eq} + \frac{2}{r\sqrt{3}}\Sigma_{eq} - \left(\sqrt{\Sigma_0 - \Sigma_{eq}^2}\right)\right)\left(\sqrt{\Sigma_0 - \Sigma_{eq}^2}\right)\\ &= 2\Sigma_{eq}\frac{d}{dr}\Sigma_{eq} + \left(2p\Sigma_0^2\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\frac{3}{2\Sigma_0}\frac{d}{dr}\Sigma_m\right) - 2p\sinh\left(\frac{3}{2}\frac{\Sigma_m}{\Sigma_0}\right)\end{aligned} \tag{1.90}$$

Let $\alpha = 3\sqrt{3}$, $\gamma = \frac{3}{2\Sigma_0}$, and $\rho = p\Sigma_0$ then Eq.(1.90) reads

$$\begin{aligned}&\left(\frac{d}{dr}\Sigma_{eq}+\frac{6}{r}\Sigma_{eq}-\alpha\left(\sqrt{\Sigma_0-\Sigma_{eq}^2}\right)\right)\left(\sqrt{\Sigma_0-\Sigma_{eq}^2}\right)\\&=2\Sigma_{eq}\frac{d}{dr}\Sigma_{eq}+\left(\rho\sinh\left(\gamma\Sigma_m\right)\frac{d}{dr}\Sigma_m\right)-2p\sinh\left(\gamma\Sigma_m\right)\end{aligned}\tag{1.91}$$

Combining like terms in Eq.(1.91) gives us the result

$$\begin{aligned}&\left(\frac{d}{dr}\Sigma_{eq}+\frac{6}{r}\Sigma_{eq}-\alpha\left(\sqrt{\Sigma_0-\Sigma_{eq}^2}\right)\right)\left(\sqrt{\Sigma_0-\Sigma_{eq}^2}\right)\\&=2\Sigma_{eq}\frac{d}{dr}\Sigma_{eq}+\sinh\left(\gamma\Sigma_m\right)\left(\rho\frac{d}{dr}\Sigma_m-2p\right)\end{aligned}\tag{1.92}$$

Factoring the left side of Eq.(1.92) and combining terms yields

$$\begin{aligned}\frac{d}{dr}\Sigma_{eq}\left(\sqrt{\Sigma_0-\Sigma_{eq}^2}-2\Sigma_{eq}\right)+\frac{6}{r}\Sigma_{eq}\left(\sqrt{\Sigma_0-\Sigma_{eq}^2}\right)-\alpha\left(\Sigma_0-\Sigma_{eq}^2\right)\\=\sinh\left(\gamma\Sigma_m\right)\left(\rho\frac{d}{dr}\Sigma_m-2p\right)\end{aligned}\tag{1.93}$$

Since Σ_{eq} and Σ_m are independent of one another both sides of Eq.(1.93) must be equal to some constant value, say λ, then we can solve the system of ODE

$$\frac{d}{dr}\Sigma_{eq}\left(\sqrt{\Sigma_0-\Sigma_{eq}^2}-\beta\Sigma_{eq}\right)+\frac{6}{r}\Sigma_{eq}\left(\sqrt{\Sigma_0-\Sigma_{eq}^2}\right)-\alpha\left(\Sigma_0-\Sigma_{eq}^2\right)=\lambda$$

$$\rho\sinh\left(\gamma\Sigma_m\right)\frac{d}{dr}\Sigma_m-2p\sinh\left(\gamma\Sigma_m\right)=\lambda$$

where the value λ is to be determined later. By rearranging the above equation we get

$$\frac{\rho\sinh\left(\gamma\Sigma_m\right)}{\lambda+2p\sinh\left(\gamma\Sigma_m\right)}d\Sigma_m=dr\tag{1.94}$$

Integrating both sides of Eq.(1.94) we have

$$\int\frac{\rho\sinh\left(\gamma\Sigma_m\right)}{\lambda+2p\sinh\left(\gamma\Sigma_m\right)}d\Sigma_m=r+C_1\tag{1.95}$$

for some constant C_1. The integral on the left is complicated but nonetheless can be evaluated by analytical means. Applying linearity Eq.(1.95) writes

$$\int\frac{\rho\sinh\left(\gamma\Sigma_m\right)}{\lambda+2p\sinh\left(\gamma\Sigma_m\right)}d\Sigma_m=\rho\int\frac{\sinh\left(\gamma\Sigma_m\right)}{\lambda+2p\sinh\left(\gamma\Sigma_m\right)}d\Sigma_m\tag{1.96}$$

Next let $u = \gamma\Sigma_m$ then $\dfrac{du}{d\Sigma_m} = \gamma$ so $\dfrac{du}{\gamma} = d\Sigma_m$. Therefore in terms of u Eq.(1.96) writes

$$\rho \int \frac{\sinh(\gamma\Sigma_m)}{\lambda + 2p\sinh(\gamma\Sigma_m)} d\Sigma_m = \frac{\rho}{\gamma}\int \frac{\sinh(u)}{\lambda + 2p\sinh(u)} du \tag{1.97}$$

Write $\sinh(u)$ as $\sin(u) = \dfrac{1}{2p}(2p\sinh(u) + \lambda) - \dfrac{\lambda}{2p}$ then

$$\begin{aligned}\int \left(\frac{\sinh(u)}{2p\sinh(u) + \lambda}\right) du &= \int \left(\frac{\frac{1}{2p}(2p\sinh(u) + \lambda)}{2p\sinh(u) + \lambda} - \frac{\frac{\lambda}{2p}}{2p\sinh(u) - \lambda}\right) du \\ &= \frac{u}{2p} - \frac{\lambda}{2p}\int \frac{1}{2p\sinh(u) + \lambda} du\end{aligned} \tag{1.98}$$

Now we solve

$$\int \left(\frac{1}{2p\sinh(u) + \lambda}\right) du.$$

By the tangent half angle substitution, otherwise known as the Weierstrass substitution, the integral writes

$$\int \left(\frac{1}{2p\sinh(u) + \lambda}\right) du = \int \left(\frac{1}{\frac{4p\tanh\left(\frac{u}{2}\right)}{1 - \tanh^2\left(\frac{u}{2}\right)} + \lambda}\right) du \tag{1.99}$$

Substitute $\kappa = \tanh\left(\frac{u}{2}\right)$ so that $\dfrac{d\kappa}{du} = \dfrac{\operatorname{sech}^2\left(\frac{u}{2}\right)}{2}$, $du = \dfrac{2}{\operatorname{sech}^2\left(\frac{u}{2}\right)}$, and $d\kappa = \dfrac{2}{1-\kappa^2} du$ we obtain

$$\begin{aligned}\int \left(\frac{1}{\frac{4p\tanh\left(\frac{u}{2}\right)}{1 - \tanh^2\left(\frac{u}{2}\right)} + \lambda}\right) du &= -2\int \frac{1}{\lambda\kappa^2 - 4p\kappa - \lambda} d\kappa \\ &= -2\int \frac{d\kappa}{\left(\sqrt{\lambda}\kappa - \frac{2p}{\sqrt{\lambda}}\right)^2 - \frac{4p^2}{\lambda} - \lambda}\end{aligned} \tag{1.100}$$

Next, let $w = \dfrac{\lambda\kappa - 2p}{\sqrt{\lambda}\sqrt{-\dfrac{4p^2}{\lambda} - \lambda}}$ then $\dfrac{dw}{dv} = \dfrac{\sqrt{\lambda}}{-\dfrac{4p^2}{\lambda} - \lambda}$ so $dw\dfrac{-\dfrac{4p^2}{\lambda} - \lambda}{\sqrt{\lambda}} = dv$. Upon substitution we get

$$
\begin{aligned}
-2\int \frac{d\kappa}{\left(\sqrt{\lambda}\kappa - \dfrac{2p}{\sqrt{\lambda}}\right)^2 - \dfrac{4p^2}{\lambda} - \lambda} &= -2\int \frac{\sqrt{\dfrac{-4p^2}{\lambda} - \lambda}}{\sqrt{\lambda}\left(\left(-\dfrac{4p^2}{\lambda} - \lambda\right) w^2 - \dfrac{4p^2}{\lambda} - \lambda\right)} dw \\
&= \left(\frac{1}{\dfrac{1}{\sqrt{\lambda}}\sqrt{-\dfrac{4p^2}{\lambda} - \lambda}}\right) \int \frac{1}{w^2 + 1} dw \\
&= \left(\frac{1}{\dfrac{1}{\sqrt{\lambda}}\sqrt{-\dfrac{4p^2}{\lambda} - \lambda}}\right) \arctan(w)
\end{aligned}
\tag{1.101}
$$

After plugging in $w = \dfrac{\lambda\kappa - 2p}{\sqrt{\lambda}\sqrt{-\dfrac{4p^2}{\lambda} - \lambda}}$ we find

$$
-2\int \frac{d\kappa}{\left(\sqrt{\lambda}\kappa - \dfrac{2p}{\sqrt{\lambda}}\right)^2 - \dfrac{4p^2}{\lambda} - \lambda} = \frac{\arctan\left(\dfrac{\lambda \tanh\left(\frac{u}{2}\right) - 2p}{\sqrt{\lambda}\sqrt{-\dfrac{4p^2}{\lambda} - \lambda}}\right)}{\sqrt{\lambda}\sqrt{-\dfrac{4p^2}{\lambda} - \lambda}} \tag{1.102}
$$

Upon substitution $u = \gamma \varSigma_m$ the integral is solved as

$$
\int \frac{\rho \sinh(\gamma \varSigma_m)}{\lambda + 2p \sinh(\gamma \varSigma_m)} d\varSigma_m = \frac{\rho\left(\frac{\varSigma_m}{p} - \frac{\lambda \ln\left(\frac{\left|2p\mathrm{e}^{-\gamma \varSigma_m} - \sqrt{4p^2+\lambda^2} - \lambda\right|}{\left|2p\mathrm{e}^{-\gamma \varSigma_m} + \sqrt{4p^2+\lambda^2} - \lambda\right|}\right)}{\gamma p \sqrt{4p^2+\lambda^2}}\right)}{2} \tag{1.103}
$$

Therefore, by Eq.(1.95) we find $\varSigma_m$ satisfies the implicit expression

$$\frac{\rho\left(\frac{\Sigma_m}{p}-\frac{\lambda\ln\left(\frac{\left|2pe^{-\gamma\Sigma_m}-\sqrt{4p^2+\lambda^2}-\lambda\right|}{\left|2pe^{-\gamma\Sigma_m}+\sqrt{4p^2+\lambda^2}-\lambda\right|}\right)}{\gamma p\sqrt{4p^2+\lambda^2}}\right)}{2}=r+C_1 \tag{1.104}$$

We solve λ by solving the root of the equation

$$\begin{aligned}&\frac{\Sigma_m(r_e)-\Sigma_m(r_i)}{p}-\frac{2\left(r_i-r_e\right)}{\rho}\\&-\frac{\lambda\ln\left(\frac{\left|2pe^{-\gamma\Sigma_m(r_e)}-\sqrt{4p^2+\lambda^2}-\lambda\right|}{\left|2pe^{-\gamma\Sigma_m(r_e)}+\sqrt{4p^2+\lambda^2}-\lambda\right|}\cdot\frac{\left|2pe^{-\gamma\Sigma_m(r_i)}-\sqrt{4p^2+\lambda^2}-\lambda\right|}{\left|2pe^{-\gamma\Sigma_m(r_i)}+\sqrt{4p^2+\lambda^2}-\lambda\right|}\right)}{\gamma p\sqrt{4p^2+\lambda^2}}\end{aligned} \tag{1.105}$$