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Numerical implementation for a constitutive model of plastic behavior of metals undergoing phase transformation incorporating the mixed isotropic-kinematic hardening

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1. INTRODUCTION

Some new needs have recently appeared concerning the possibility of a mixed work hardening isotropic-kinematic in the modeling of the plastic behavior of metals during phase transformation developed by Leblond et al. [1, 2]. The objective of this note is to describe without insisting on the theoretical aspects, such a modeling, as well as its numerisation within the framework of the finite element element analysis of phase transformations in metals.

This opportunity will be taken to simplify and rationalize the numerization of this behavior for the other types of hardening (namely, ideal perfect plasticity, isotropic hardening, kinematic hardening). First, in fact, this mumerical impementation presents some unnecessary complications, such as the use sometimes of a semi-implicit algorithm whereas a totally explicit, much simpler, algorithm does not lead to a significant degradation of the precision. Second, various additional effects have been introduced into the modeling (for example, restoration or the memory of work hardening during transformations, effect of large transformations, etc.), their numerical implementation not always being carried out in the same mind than the initial numerization (usually for the sake of simplicity). A general "grooming" therefore seems desirable.

Finally, let us point out that as much as this "grooming" leads to quite numerous (if not profound) modifications to the program, the consequences for the user are minimal. In the case of perfect plasticity or pure isotropic or pure kinematic work hardening, the setting in data is strictly unchanged. In the case of mixed work hardening, the data setting, with regard to work hardening, is the same as in the absence of metallurgy.

2. THERMO-PLASTICITY BEHAVIOR FOR A MIXED ISOTROPIC-KINEMATIC HARDENING.

Let us begin for the sake of completeness by recalling the model of plastic behavior with mixed work hardening used in the standard finite element codes, in the absence of phase transformation. We consider the general case of a variable temperature and large deformations.

Let $\sigma_0(T)$ be the initial limit of elasticity, before work hardening, function only of the temperature$T$. Let $\sigma(\varepsilon_{eq}, T)$ be the stress observed in an initial tensile test at the temperature T, function of this temperature and of the cumulated plastic deformation ε_{eq}. Let

$$\overline{\sigma}(\varepsilon_{eq}, T) \equiv \sigma(\varepsilon_{eq}, T) - \sigma_0(T) \tag{1}$$

the part of this stress coming from work hardening. Finally, let p be the proportion of work hardening which is of an isotropic nature.

The limit of elasticity is therefore

$$\sigma^Y(\varepsilon_{eq}, T) \equiv \sigma_0(T) + p\overline{\sigma}(\varepsilon_{eq}, T) \tag{2}$$

The yield criterion is then written as

$$\sigma_{eq} \equiv \left[\frac{3}{2}(\underline{\underline{s}} - \underline{\underline{a}}) : (\underline{\underline{s}} - \underline{\underline{a}})\right]^{\frac{1}{2}} \leq \sigma^Y(\varepsilon_{eq}, T) \tag{3}$$

where $\underline{\underline{s}}$ denotes the deviatoric stress. The evolution equation of the center $\underline{\underline{a}}$ of the domain of elasticity is:

$$\check{\underline{\underline{a}}} \equiv \dot{\underline{\underline{a}}} + \left(\dot{\underline{\underline{a}}}\right)_{GT} = \frac{2}{3}(1-p)\frac{\partial\overline{\sigma}}{\partial\varepsilon_{eq}}(\varepsilon_{eq}, T)\underline{\underline{d}}^p + \frac{1}{\overline{\sigma}}\frac{\partial\overline{\sigma}}{\partial T}(\varepsilon_{eq}, T)\underline{\underline{a}}\dot{T} \tag{4}$$

In this equation, $\check{\underline{\underline{a}}}$ denotes the objective derivative of a chosen (for example, those of Jaumann or Molinari) and $\left(\dot{\underline{\underline{a}}}\right)_{GT}$ the part, in the expression of this objective derivative, due to large deformations. In addition, $\underline{\underline{d}}^p$ denotes the plastic strain rate (Eulerian). Finally, for the record, the plastic constitutive law is the same as usual.

$$\underline{\underline{d}}^p = \frac{3}{2}\frac{\dot{\varepsilon_{eq}}}{\sigma_{eq}}(\underline{\underline{s}} - \underline{\underline{a}}), \qquad \dot{\varepsilon}_{eq} = \left(\frac{2}{3}\underline{\underline{d}}^p : \underline{\underline{d}}^p\right)^{\frac{1}{2}} \tag{5}$$

3. PLASTIC BEHAVIOR DURING PHASE TRANSFORMATION IN THE CASE OF MIXED ISOTROPIC-KINEMATIC HARDENING

We will not include here the details to the homogenization approach leading to the macroscopic equations of plastic behavior during phase transformations. We will only indicate the results, within the framework of a mixed isotropic-kinematic hardening.

3.1. Generalities

The parent-phase (γ) is denoted with an index of 1, and the daughter-phase α with an index of 2; z denotes the proportion of daughter-phase ($\dot{z}$). We denote $\overline{\sigma}_i^{(}\varepsilon_i^{eff}, T)$ the part coming from the work hardening in the stress observed in a simple tensile test, carried out on a sample of pure phase i. This quantity is a function of the effective plastic strain eff of the phase i, which may differ from the equivalent strain due to the phenomena of memory and restoration of work hardening during the transformations. We denote $\sigma_i^Y(\varepsilon_i^{eff}, T)$ the limit of elasticity of phase i, given by a formula analogous to Eq.2 (with σ_i^0 and $\overline{\sigma}_i$ instead of $\overline{\sigma}_i$ and σ_0). We denote $\underline{\underline{a}}_i$ the center of the elasticity domain of phase i.

Finally, the overall limit stress is given by the formula.

$$\sigma^Y(\varepsilon_1^{eff}, \varepsilon_2^{eff}, z, T) = [1 - f(z)]\sigma_1^Y(\varepsilon_1^{eff}, T) + f(z)\sigma_2^Y(\varepsilon_2^{eff}, T) \tag{6}$$

3.2. Case where the stress is less than the yield limit

This case is defined by the condition

$$\sigma_{eq} < \sigma^Y, \quad \sigma_{eq} \equiv \left[\frac{3}{2}(\underline{\underline{s}} - \underline{\underline{a}}) : (\underline{\underline{s}} - \underline{\underline{a}})\right]^{\frac{1}{2}}, \quad \underline{\underline{a}} \equiv (1 - z)\underline{\underline{a}}_1 + z\underline{\underline{a}}_2 \tag{7}$$

The other part of the plastic strain rate corresponding to the plasticity of transformation is written as

$$\underline{\underline{d}}^{pt} = -3\frac{\varepsilon_2^{th}(T) - \varepsilon_1^{th}(T)}{\sigma_1^Y(\varepsilon_1^{eff}, T)} h\left(\frac{\sigma_{eq}}{\sigma^Y}\right)(\ln z)(\underline{\underline{s}} - \underline{\underline{a}}_1)\dot{z} \tag{8}$$

where $\varepsilon_i^{th}(T)$ is the thermal deformation of the phase i. The part of the rate of plastic deformation corresponding to the plastic plasticity is decomposed into 2 terms, one, $\underline{\underline{d}}_\sigma^{pc}$ coming from the variations of $\underline{\underline{\sigma}}$ and the other, $\underline{\underline{d}}_T^{pc}$, coming from the variations of T is given by

$$\underline{\underline{d}}_\sigma^{pc} = \frac{3}{2}\frac{1 - z}{\sigma_1^Y(\varepsilon_1^{eff}, T)}\frac{g(z)}{E}(\underline{\underline{s}} - \underline{\underline{a}}_1)(\dot{\sigma}_1^{eq})_s \tag{9}$$

$$(\dot{\sigma}_1^{eq})_s \equiv \frac{3}{2\sigma_1^{eq}}(\underline{\underline{s}} - \underline{\underline{a}}_1) : \check{\underline{\underline{s}}}, \quad \sigma_1^{eq} \equiv \left[\frac{3}{2}(\underline{\underline{s}} - \underline{\underline{a}}_1) : (\underline{\underline{s}} - \underline{\underline{a}}_1)\right]^{\frac{1}{2}} \tag{10}$$

$$\underline{\underline{d}}_T^{pc} = 3\frac{\alpha_1 - \alpha_2}{\sigma_1^Y(\varepsilon_1^{eff}, T)} z(\ln z)(\underline{\underline{s}} - \underline{\underline{a}}_1)\dot{T} \tag{11}$$

where α_i denotes the coefficient of the thermal dilatation of the phase i.

The evolution equations of the effective plastic deformation of the phases are as follow

$$\dot{\varepsilon}_1^{eff} = -2\frac{\varepsilon_1^{th}(T) - \varepsilon_2^{th}(T)}{1-z} h\left(\frac{\sigma_{eq}}{\sigma^Y}\right)(\ln z)\dot{z} + \frac{g(z)}{E}(\dot{\sigma}_1^{eq})_s + 2\frac{\alpha_1 - \alpha_2}{1-z} z(\ln z)\dot{T} \tag{12}$$

$$\dot{\varepsilon}_2^{eff} = \frac{\dot{z}}{z}\varepsilon_2^{eff} + \theta\frac{\dot{z}}{z}\dot{\varepsilon}_1^{eff} \tag{13}$$

where θ denotes the memory coefficient of the work hardening during the transformation ($\theta = 0$ means that the hardening of the mother-phase is completely forgotten by the daughter-phase during the transformation, $\theta = 1$, that this work hardening is, on the contrary, entirely transferred to the daughter-phase.) Finally, the evolution equations of the centers of the elasticity domain of the phases are as follows:

$$\check{\underline{\underline{a}}}_1 \equiv \dot{\underline{\underline{a}}}_1 + (\dot{\underline{\underline{a}}}_1)_{GT} = \frac{2}{3}\frac{1-p}{1-z}\frac{\partial\overline{\sigma_1}}{\partial\varepsilon_1^{eff}}(\varepsilon_1^{eff}, T)(\underline{\underline{d}}^{pt} + \underline{\underline{d}}_\sigma^{pc} + \underline{\underline{d}}_T^{pc}) + \frac{1}{\overline{\sigma}_1}\frac{\partial\overline{\sigma_1}}{\partial T}(\varepsilon_1^{eff}, T)\underline{\underline{a}}_1\dot{T} \tag{14}$$

$$\check{\underline{\underline{a}}}_2 \equiv \dot{\underline{\underline{a}}}_2 + (\dot{\underline{\underline{a}}}_2)_{GT} = -\frac{\dot{z}}{z}\underline{\underline{a}}_2 + \theta\frac{\dot{z}}{z}\underline{\underline{a}}_1 + \frac{1}{\overline{\sigma}_2}\frac{\partial\overline{\sigma_2}}{\partial T}(\varepsilon_2^{eff}, T)\underline{\underline{a}}_2\dot{T} \tag{15}$$

3.3. Case where the stress equals the yield limit

This case is defined by the condition

$$\sigma_{eq} = \sigma^Y \tag{16}$$

where σ_{eq} is always defined by by the relations Eq.(7). The flow rule is then

$$\underline{\underline{d}}^p = \frac{3}{2}\frac{\dot{\varepsilon}_{eq}}{\sigma_{eq}}(\underline{\underline{s}} - \underline{\underline{a}}) \qquad (\text{with } \dot{\varepsilon}_{eq} = \left(\frac{2}{3}\underline{\underline{d}}_p : \underline{\underline{d}}_p\right)^{\frac{1}{2}}) \tag{17}$$

The evolution equations of the work hardening are written as follows:

$$\dot{\varepsilon}_1^{eff} = \dot{\varepsilon}_{eq} \tag{18}$$

$$\dot{\varepsilon}_2^{eff} = \dot{\varepsilon}_{eq} - \frac{\dot{z}}{z}\varepsilon_2^{eff} + \theta\frac{\dot{z}}{z}\dot{\varepsilon}_1^{eff} \tag{19}$$

$$\check{\underline{\underline{a}}}_1 = \frac{2}{3}(1-p)\frac{\partial\overline{\sigma_1}}{\partial\varepsilon_1^{eff}}(\varepsilon_1^{eff}, T)\underline{\underline{d}}^p + \frac{1}{\overline{\sigma}_1}\frac{\partial\overline{\sigma_1}}{\partial T}(\varepsilon_1^{eff}, T)\underline{\underline{a}}_1\dot{T} \tag{20}$$

$$\check{\underline{\underline{a}}}_2 = \frac{2}{3}(1-p)\frac{\partial\overline{\sigma_2}}{\partial\varepsilon_2^{eff}}(\varepsilon_2^{eff}, T)\underline{\underline{d}}^p + \frac{1}{\overline{\sigma}_2}\frac{\partial\overline{\sigma_2}}{\partial T}(\varepsilon_2^{eff}, T)\underline{\underline{a}}_2\dot{T} - \frac{\dot{z}}{z}\underline{\underline{a}}_2 + \theta\frac{\dot{z}}{z}\underline{\underline{a}}_1 \tag{21}$$

4. NUMERICAL IMPLEMENTATION

About each equation arises the problem of the choice of the algorithm: explicit, implicit, semi implicit. The choices made here, which do not coincide exactly with the previous choice, are dictated by the following considerations.

a. An explicit algorithm is preferable if it makes it possible to simplify the digitization without significantly degrading its accuracy;
b. an implicit algorithm is preferable with respect to the direction of the plastic flow (given by the stress deviator) for the sake of consistency with the standard programming in finite element codes;
c. a semi-implicit algorithm is preferable if it significantly improves accuracy, or if, even if it doesn't, it doesn't significantly complicate the numerization.

4.1. Case where the yield limit is not reached

The partition of the deviator of the increment of the total strain (thermal part substracted) between the times t and $t+\Delta t$ is written as

$$\Delta \underline{\underline{e}} = \Delta \underline{\underline{e}}^e + \Delta \underline{\underline{\varepsilon}}^p = \Delta \underline{\underline{e}}^e + (\Delta \underline{\underline{\varepsilon}}^p)' + (\Delta \underline{\underline{\varepsilon}}^p)'' \tag{22}$$

where the term $\left((\Delta \underline{\underline{\varepsilon}}^p)'\right)$ corresponds to $\left(\underline{\underline{d}}_{pt} + \underline{\underline{d}}_T^{pc}\right)\Delta t$ and $\left((\Delta \underline{\underline{\varepsilon}}^p)''\right)$ to $\left(\underline{\underline{d}}_\sigma^{pc}\right)\Delta t$. The expressions of these terms are the following, where F denotes the function of the von Mises $\left(F(\underline{\underline{X}}) = \left(\frac{3}{2}\underline{\underline{X}} : \underline{\underline{X}}\right)^{1/2}\right)$:

$$(\Delta \underline{\underline{\varepsilon}}^p)' = \frac{A}{2}\left[1 + \frac{F(\underline{\underline{s}} - \underline{\underline{a}}_1)}{F(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1)}\right](\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1) \tag{23}$$

$$(\Delta \underline{\underline{\varepsilon}}^p)'' = \frac{B}{2}\left[1 + \frac{F(\underline{\underline{s}} - \underline{\underline{a}}_1)}{F(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1)}\right](\Delta\sigma_1^{eq})_s(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1) \tag{24}$$

In the expression Eq.(23), A is given by

$$\begin{aligned} A = & -3\frac{\varepsilon_2^{th}(T) - \varepsilon_1^{th}(T) + \varepsilon_2^{th}(T+\Delta T) - \varepsilon_1^{th}(T+\Delta T)}{\sigma_1^Y(\varepsilon_1^{eff}, T) + \sigma_1^Y(\varepsilon_1^{eff}, T+\Delta T)} h\left(\frac{\sigma_{eq}}{\sigma^Y}\right)\{(z+\Delta z)[\ln(z+\Delta z) - 1] - z(\ln z - 1)\} \\ & + 3\frac{\varepsilon_1^{th}(T+\Delta T) - \varepsilon_1^{th}(T) + \varepsilon_2^{th}(T+\Delta T) - \varepsilon_2^{th}(T)}{\sigma_1^Y(\varepsilon_1^{eff}, T) + \sigma_1^Y(\varepsilon_1^{eff}, T+\Delta T)}[z\ln z + (z+\Delta Z)\ln(z+\Delta z)] \end{aligned} \tag{25}$$

The term $h(\sigma_{eq}\bar{/}\sigma^Y)$ in this expression is discretized explicitly. Moreover, the term comes from an exact integration of $\ln(z)$ between z and $z+\Delta z$ in Eq.(8), the other terms being considered constant. Numerical experiments have shown the importance of such exact integration to conveniently reproduce stress dilatometry tests.

The quantity B in Eq.(24) is given by

$$B = 3\frac{(1-z)g(z) + (1-z-\Delta z)g(z+\Delta z)}{\left[\sigma_1^Y(\varepsilon_1^{eff}, T) + \sigma_1^Y(\varepsilon_1^{eff}, T+\Delta T)\right][E(T) + E(T+\Delta T)]} \tag{26}$$

In addition, $(\Delta\sigma_1^{eq})_s$, is given by

$$(\Delta\sigma_1^{eq})_s = \frac{3}{2F(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1)}(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1) : (\Delta\underline{\underline{s}})_{OBJ} \tag{27}$$

where $(\Delta\underline{\underline{s}})_{OBJ}\left(\equiv \check{\underline{\underline{s}}}\right)$ is the objective part of the deviatoric stress rate

$$(\Delta\underline{\underline{s}})_{OBJ} \equiv \Delta\underline{\underline{s}} + (\Delta\underline{\underline{s}})_{GT} \tag{28}$$

The hypo-elasticity law is given by

$$(\Delta\underline{\underline{s}})_{OBJ}(= \Delta\underline{\underline{s}} + (\Delta\underline{\underline{s}})_{GT}) = 2\mu\Delta\underline{\underline{e}}^e + (\Delta\underline{\underline{s}})_T \tag{29}$$

where μ denotes the shear coefficient at the time $t + \Delta t$ (this notation is used here rather than the more logical notation $\mu + \Delta\mu$ to simplify the writing) and $(\Delta\underline{\underline{s}})_T$ the part of $\Delta\underline{\underline{s}}$ coming from the variation of the temperature (via its influence on μ.) The evolution equation of ε_1^{eff} is discretized as the following equation:

$$\Delta\varepsilon_1^{eff} = \frac{2}{3}\frac{\sigma_1^Y(\varepsilon_1^{eff}, T) + \sigma_1^Y(\varepsilon_1^{eff}, T+\Delta T)}{(1-z) + (1-z+\Delta z)}[A + B(\Delta\sigma_1^{eq})_s] \tag{30}$$

The equation of ε_1^{eff} is written in the form $\frac{d}{dt}\left(0z\varepsilon_1^{eff}\right) = \theta\varepsilon_1^{eff}\dot{z}$ before being discrtized by

$$\Delta(z\varepsilon_2^{eff}) \equiv (z+\Delta z)(\varepsilon_2^{eff} + \Delta\varepsilon_2^{eff}) - z\varepsilon_2^{eff} = \theta\varepsilon_1^{eff}\Delta z \tag{31}$$

Similarly, the evolution equations of $\underline{\underline{a}}_1$ and $\underline{\underline{a}}_2$ are discretized as follows:

$$\begin{aligned}
&(\Delta\underline{\underline{a}}_1)_{OBJ} \equiv \Delta\underline{\underline{a}}_1 + (\Delta\underline{\underline{a}}_1)_{GT} = \frac{2}{3}\frac{1-p}{1-z-\Delta z/2}\frac{\partial\overline{\sigma}_1}{\partial\varepsilon_1^{eff}}(\varepsilon_1^{eff}, T+\Delta T) \\
&\left[(\Delta\underline{\underline{\varepsilon}}^p)' + \varepsilon^p)''\right] + (\Delta\underline{\underline{a}}_1)_T \\
&\implies \Delta\underline{\underline{a}}_1 = \frac{2}{3}\frac{1-p}{1-z-\Delta z/2}\frac{\partial\overline{\sigma}_1}{\partial\varepsilon_1^{eff}}(\varepsilon_1^{eff}, T+\Delta T)\left[(\Delta\underline{\underline{\varepsilon}}^p)' + (\Delta\underline{\underline{\varepsilon}}^p)''\right] \\
&-(\Delta\underline{\underline{a}}_1)_{GT} + (\Delta\underline{\underline{a}}_1)_T
\end{aligned} \tag{32}$$

Note in Eq.(32) the use of the hardening slope $\frac{\partial\overline{\sigma}_1}{\partial\varepsilon_1^{eff}}(\varepsilon_1^{eff}, T+\Delta T)$ instead of the secant as previously. The interest of this replacement is to lead to an explicit resolution not requiring iterations on the parameter of work hardening $\varepsilon_1^{eff} + \Delta\varepsilon_1^{eff}$; it is licit insofar as there are no criteria to be satisfied exactly at the time $t + \Delta t$ (it will not be the same if the yield limit is reached.) Moreover, note that the terms $(\Delta\underline{\underline{a}}_i)_{GT}$ and $(\Delta\underline{\underline{a}}_i)_T$ are discretized in an explicit way and therefore known from the beginning.

Now let us move on to solving these equations; the principal unkowns used are

$$\Delta(z\underline{\underline{a}}_2) \equiv (z+\Delta z)(\underline{\underline{a}}_2 + \Delta\underline{\underline{a}}_2) - z\underline{\underline{a}}_2 = \theta\underline{\underline{a}}_1\Delta z - z(\Delta\underline{\underline{a}}_2)_{GT} + z(\Delta\underline{\underline{a}}_2)_T \tag{33}$$

Combining Eq.(22) and Eq.(29) we get

$$X = F(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1), \quad Y = (\Delta\sigma_1^{eq})_s \tag{34}$$

$$\begin{aligned} &\Delta\underline{\underline{s}} = 2\mu\Delta\underline{\underline{e}}^e - (\Delta\underline{\underline{s}})_{GT} + (\Delta\underline{\underline{s}})_T \quad \Longrightarrow \\ &\underline{\underline{s}} + \Delta\underline{\underline{s}} \equiv (\underline{\underline{s}} + \Delta\underline{\underline{s}})^{el} - 2\mu\left[(\Delta\underline{\underline{\varepsilon}}^p)' + (\Delta\underline{\underline{\varepsilon}}^p)''\right] \end{aligned} \tag{35}$$

$$(\underline{\underline{s}} + \Delta\underline{\underline{s}})^{el} \equiv \underline{\underline{s}} + 2\mu\Delta\underline{\underline{e}} - (\Delta\underline{\underline{s}})_{GT} + (\Delta\underline{\underline{s}})_T \tag{36}$$

where $(\underline{\underline{s}} + \Delta\underline{\underline{s}})^{el}$, known quantity, is the deviatoric stress at $t + \Delta t$ elastically calculted, that is by considering the deviatoric part of the increment of the total strain $\Delta\underline{\underline{e}}$ (with the termal part not being accounted for) as purely elastic. Adding $-\underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1$ to the two sides of Eq.(35) and taking into account Eq.(32), we get

$$\begin{aligned} \underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1 &= (\underline{\underline{s}} + \Delta\underline{\underline{s}})^{el} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1 - 2\mu\left[(\Delta\underline{\underline{\varepsilon}}^p)' + (\Delta\underline{\underline{\varepsilon}}^p)''\right] \\ &= (\underline{\underline{s}} + \Delta\underline{\underline{s}})^{el} - \underline{\underline{a}}_1 + (\Delta\underline{\underline{a}}_1)_{GT} - (\Delta\underline{\underline{a}}_1)_T \\ &\quad - \left[2\mu + \frac{2}{3}\frac{1-p}{1-z-\Delta z/2}\frac{\partial\overline{\sigma}_1}{\partial\varepsilon_1^{eff}}(\varepsilon_1^{eff}, T+\Delta T)\right]\left[(\Delta\underline{\underline{\varepsilon}}^p)' + (\Delta\underline{\underline{\varepsilon}}^p)''\right] \end{aligned}$$

which, by setting

$$\underline{\underline{s}}^* \equiv (\underline{\underline{s}} + \Delta\underline{\underline{s}})^{el} - \underline{\underline{a}}_1 + (\Delta\underline{\underline{a}}_1)_{GT} - (\Delta\underline{\underline{a}}_1)_T \tag{37}$$

is equivalent to

$$H \equiv \frac{1-p}{1-z-\Delta z/2}\frac{\partial\overline{\sigma}_1}{\partial\varepsilon_1^{eff}}(\varepsilon_1^{eff}, T+\Delta T) \tag{38}$$

(these quantities are known):

$$\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1 = \underline{\underline{s}}^* - 2\left(\mu + \frac{H}{3}\right)\left[(\Delta\underline{\underline{\varepsilon}}^p)' + (\Delta\underline{\underline{\varepsilon}}^p)''\right]$$

According to Eq.(23) and Eq.(24) and the notations Eq.(34) we get

$$(\Delta\underline{\underline{\varepsilon}}^p)' + (\Delta\underline{\underline{\varepsilon}}^p)'' = \frac{1}{2}(A+BY)\left(1+\frac{F(\underline{\underline{s}}-\underline{\underline{a}}_1)}{X}\right)(\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}_1-\Delta\underline{\underline{a}}_1) \tag{39}$$

which by reporting in the previous equation reads

$$\begin{aligned}&\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}_1-\Delta\underline{\underline{a}}_1 = \underline{\underline{s}}^* - \left(\mu+\frac{H}{3}\right)(A+BY)\left(1+\frac{F(\underline{\underline{s}}-\underline{\underline{a}}_1)}{X}\right)(\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}_1-\Delta\underline{\underline{a}}_1)\\ &\Rightarrow \left[1+\left(\mu+\frac{H}{3}\right)(A+BY)\left(1+\frac{F(\underline{\underline{s}}-\underline{\underline{a}}_1)}{X}\right)\right](\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}_1-\Delta\underline{\underline{a}}_1) = \underline{\underline{s}}^*\end{aligned} \tag{40}$$

This equation implies that the (unknown) tensor $\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}_1-\Delta\underline{\underline{a}}_1$ is positively parallel to the (unknown) tensor $\underline{\underline{s}}^*$. Thus,

$$\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}_1-\Delta\underline{\underline{a}}_1 = \frac{X}{F(\underline{\underline{s}}^*)}\underline{\underline{s}}^* \tag{41}$$

which brings the calculation of the unknown $\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}_1-\Delta\underline{\underline{a}}_1$ to the same of the norm of X. Moreover, by taking the von Mises function of Eq.(41), we obtain:

$$\begin{aligned}&X+\left(\mu+\frac{H}{3}\right)(A+BY)\left(X+F(\underline{\underline{s}}-\underline{\underline{a}}_1)\right) = F(\underline{\underline{s}}^*) \qquad \Rightarrow\\ &A+BY = \frac{F(\underline{\underline{s}}^*)-X}{(\mu+\frac{H}{3})\left(X+F(\underline{\underline{s}}-\underline{\underline{a}}_1)\right)} \quad \Leftrightarrow Y = \frac{1}{B}\left[\frac{F(\underline{\underline{s}}^*)-X}{(\mu+\frac{H}{3})\left(X+F(\underline{\underline{s}}-\underline{\underline{a}}_1)\right)} - A\right]\end{aligned} \tag{42}$$

The unknown quantity Y can now be expressed as a function of the unknown X, it remains to calculate the latter. For this, let us re-express $\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}_1-\Delta\underline{\underline{a}}_1$ using the equations Eq. 28 and Eq.(28) and Eq. (30) as well as the definition Eq. (38) as:

$$\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}_1-\Delta\underline{\underline{a}}_1 = \underline{\underline{s}} - (\Delta\underline{\underline{s}})_{GT} + (\Delta\underline{\underline{s}})_{OBJ} - \underline{\underline{a}}_1 - \frac{2}{3}H\left[(\Delta\underline{\underline{\varepsilon}}^p)' + (\Delta\underline{\underline{\varepsilon}}^p)''\right] + (\Delta\underline{\underline{a}}_1)_{GT} - (\Delta\underline{\underline{a}}_1)_T$$

which, by accounting for Eq.39 and Eq. 42, reads

$$\begin{aligned}\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}_1-\Delta\underline{\underline{a}}_1 &= \underline{\underline{s}} - (\Delta\underline{\underline{s}})_{GT} - \underline{\underline{a}}_1 + (\Delta\underline{\underline{a}}_1)_{GT} - (\Delta\underline{\underline{a}}_1)_T\\ &\quad - \frac{H}{3}(A+BY)\left(1+\frac{F(\underline{\underline{s}}-\underline{\underline{a}}_1)}{X}\right)(\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}_1-\Delta\underline{\underline{a}}_1) + (\Delta\underline{\underline{s}})_{OBJ}\\ &= \underline{\underline{s}} - (\Delta\underline{\underline{s}})_{GT} - \underline{\underline{a}}_1 + (\Delta\underline{\underline{a}}_1)_{GT} - (\Delta\underline{\underline{a}}_1)_T\\ &\quad + \frac{H\left(X-F(\underline{\underline{s}}^*)\right)}{(H+3\mu)\left(X+F(\underline{\underline{s}}-\underline{\underline{a}}_1)\right)}\left(1+\frac{F(\underline{\underline{s}}-\underline{\underline{a}}_1)}{X}\right)(\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}_1-\Delta\underline{\underline{a}}_1) + (\Delta\underline{\underline{s}})_{OBJ}\end{aligned}$$

Contracting this equation with $\frac{3}{2}\underline{\underline{s}}^*$ gives, taking into account the defintion Eq.(27) of $(\Delta\sigma_1^{eq})_s \equiv Y$ and the property Eq.(41)

$$XF(\underline{\underline{s}}^*) = P + \frac{H\left(X - F(\underline{\underline{s}}^*)\right)}{(H+3\mu)\left(X + F(\underline{\underline{s}} - \underline{\underline{a}}_1)\right)}\left(X + F(\underline{\underline{s}} - \underline{\underline{a}}_1)\right)F(\underline{\underline{s}}^*) + F(\underline{\underline{s}}^*)Y$$

where we assumed that

$$P \equiv \frac{3}{2}\left(\underline{\underline{s}} - (\Delta\underline{\underline{s}})_{GT} - \underline{\underline{a}}_1 + (\Delta\underline{\underline{a}}_1)_{GT} - (\Delta\underline{\underline{a}}_1)_T\right) : \underline{\underline{s}}^*$$

(P is a known quantity). Multiplying by $(H+3\mu)\left(X + F(\underline{\underline{s}} - \underline{\underline{a}}_1)\right)$ and accounting for Eq.(42)

$$(H+3\mu)\left(X + F(\underline{\underline{s}} - \underline{\underline{a}}_1)\right)XF(\underline{\underline{s}}^*) = P\,(H+3\mu)\left(X + F(\underline{\underline{s}} - \underline{\underline{a}}_1)\right)$$
$$+ H\left(X - F(\underline{\underline{s}}^*)\right)\left(X + F(\underline{\underline{s}} - \underline{\underline{a}}_1)\right)F(\underline{\underline{s}}^*) + \frac{F(\underline{\underline{s}}^*)}{B}\left[3\left(F(\underline{\underline{s}}^*) - X\right) - A\,(H+3\mu)\left(X + F(\underline{\underline{s}} - \underline{\underline{a}}_1)\right)\right]$$

which gives after multiplication by B and re-arrangment:

Equation 43 is missing (43)

$$LX^2 + MX + N = 0 \quad (44)$$

$$L \equiv 3\mu BF(\underline{\underline{s}}^*) \quad (45)$$

$$M \equiv 3\mu BF(\underline{\underline{s}} - \underline{\underline{a}}_1)F(\underline{\underline{s}}^*) + BHF^2(\underline{\underline{s}}^*) + A(H+3\mu)F(\underline{\underline{s}}^*) - B(H+3\mu)P \quad (46)$$

The roots of this equation are $\frac{1}{2L}\left(-M \pm \sqrt{M^2 - 4LN}\right)$. The choice of the sign in front of the radical is not obvious a priori because as much as it is clear that $L > 0$, M and N can a priori take any sign. However, in practice, the coefficient B is small. We then see from Eq.(46) that $M > 0$, the $-$ sign in front of the radical then leads to a negative root, which is impossible since the equation is greater than $X \equiv F\left(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1\,()\right) > 0$, therefore the $+$ sign that must be retained.

$$N \equiv -3F^2(\underline{\underline{s}}^*) + BHF(\underline{\underline{s}} - \underline{\underline{a}}_1)F^2(\underline{\underline{s}}^*) + A(H+3\mu)F(\underline{\underline{s}} - \underline{\underline{a}}_1)F(\underline{\underline{s}}^*) - B(H+3\mu)PF(\underline{\underline{s}} - \underline{\underline{a}}_1) \quad (47)$$

$$X = \frac{1}{2L}\left(-M + \sqrt{M^2 - 4LN}\right) \quad (48)$$

However, even with this choice of signs in front of the radical, the sign of the solution is not clear because it depends on that of N, which is not itself clear (even with B

small.) It is therefore not impossible that Eq. (48) provides a negative root. In this case, it is better to adopt another algorithm which may be less precise but certainly leads to a positive root. It suffices for this purpose to replace the expressions Eq. (23) and Eq. (24), semi-implicit with respect to the norm of $\underline{\underline{s}} - \underline{\underline{a}}_1$, by the implicit expressions:

$$(\Delta\underline{\underline{\varepsilon}}^p)' = A(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1) \tag{23'}$$

$$(\Delta\underline{\underline{\varepsilon}}^p)'' = B(\Delta\sigma_1^{eq})_s(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1) \tag{24'}$$

$$L'X^2 + M'X + N' = 0 \tag{44'}$$

We can see that to find these epressions from Eq. (23) and Eq. (24), we shall replace $F(\underline{\underline{s}} - \underline{\underline{a}}_1)$ by $F\left(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1\right) \equiv X$. We obtain therefore the same equation Eq. (44) on X as previously, but by performing this substitution in the expressions Eq. (45), Eq. (46) and Eq. (47) of L, M, N, this equation becomes

$$L' = 6\mu BF(\underline{\underline{s}}^*) \tag{45'}$$

$$M' = 2BHF^2(\underline{\underline{s}}^*) + 2A(H + 3\mu)F(\underline{\underline{s}}^*) - 2B(H + 3\mu)P \tag{46'}$$

$$N = -3F^2(\underline{\underline{s}}^*) \tag{47'}$$

The coefficients L' and N' are here clearly positve and negative, respectively; therefore the product of the roots N'/L' is negative, so that there are two roots reals, one positive and the other one negative, as desired; the positive root is

$$X = \frac{1}{2L'}\left(-M' + \sqrt{M'^2 - 4L'N'}\right) \tag{48'}$$

Once X is calculated by Eq. (48), we can deduce Y by equation Eq. (42) (eventually by replacing $F(\underline{\underline{s}} - \underline{\underline{a}}_1)$ by X), $\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1$ by Eq. (41), $(\Delta\underline{\underline{\varepsilon}}^p)' + (\Delta\underline{\underline{\varepsilon}}^p)''$ by Eq. (39) (by replacing again eventually X by $F(\underline{\underline{s}} - \underline{\underline{a}}_1)$), $\underline{\underline{s}} + \Delta\underline{\underline{s}}$ by Eq. (35). It remains to update the parameter of strain hardening. The variations of ε_2^{eff} and $\underline{\underline{a}}_2$ are given by Eq. (30) and Eq. (32). The variations of ε_2^{eff} and and $\underline{\underline{a}}_2$ are obtained from Eq. (31) and Eq. (33) which can be re-written as

$$\varepsilon_2^{eff}\Delta z + (z + \Delta z)\varepsilon_2^{eff} = \theta\varepsilon_1^{eff}\Delta z \Rightarrow \Delta\varepsilon_2^{eff} = \frac{\Delta z}{z + \Delta z}(-\varepsilon_2^{eff} + \theta\varepsilon_1^{eff}) \tag{49}$$

$$\begin{aligned} &\underline{\underline{a}}_2\Delta z + (z + \Delta z)\Delta\underline{\underline{a}}_2 = \theta\underline{\underline{a}}_1\Delta z - z(\Delta\underline{\underline{a}}_2)_{GT} + z(\Delta\underline{\underline{a}}_2)_T \Rightarrow \\ &\Delta\underline{\underline{a}}_2 = \frac{1}{z + \Delta z}\left[(-\underline{\underline{a}}_2 + \theta\underline{\underline{a}}_1)\Delta z - z(\Delta\underline{\underline{a}}_2)_{GT} + z(\Delta\underline{\underline{a}}_2)_T\right] \end{aligned} \tag{50}$$

(Let us note that due the discretization explicit of $\Delta\underline{\underline{a}}_2)_{GT}$ and $(\Delta\underline{\underline{a}}_2)_T$, the variations of ε_2^{eff} and $\underline{\underline{a}}_2$ can, in fact, be calculated at the beginning, before the calculation of X and Y.)

$$\sigma_{eq} + \Delta\sigma_{eq} \equiv F(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}}) < \sigma^Y + \Delta\sigma^Y \tag{51}$$

It is finally necessary to verify the stress-limit condition not reached. defining the cas considered. The calculation of $\sigma^Y + \Delta\sigma^Y$ is immedait knowing $\varepsilon_1^{eff} + \Delta\varepsilon_1^{eff}$, $\varepsilon_2^{eff} + \Delta\varepsilon_2^{eff}$, $z + \Delta z$, $T + \Delta T$. Finding the value of $\sigma_{eq} + \Delta\sigma_{eq}$ necessitates to evaluate $(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}}$. We obtain:

$$\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}} = \underline{\underline{s}} + \Delta\underline{\underline{s}} - (1 - z - \Delta z)(\underline{\underline{a}}_1 + \Delta\underline{\underline{a}}_1) - (z + \Delta z)(\underline{\underline{a}}_2 + \Delta\underline{\underline{a}}_2)$$

All tensors being known here, we deduce $(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}}$. However, we can calculate this expression before evaluating $\Delta\underline{\underline{a}}_1$ using X, Y and the tensors known a priori $\underline{\underline{s}}^\star$, $\left(\Delta\underline{\underline{a}}_1\right)_{GT}$, $\left(\Delta\underline{\underline{a}}_1\right)_T$ and $\Delta\underline{\underline{a}}_2$. Indeed, from Eq.(32) and Eq.(39),

$$\underline{\underline{a}}_1 + \Delta\underline{\underline{a}}_1 = \underline{\underline{a}}_1 + \frac{H}{3}(A + BY)\left(1 + \frac{F(\underline{\underline{s}} - \underline{\underline{a}}_1)}{X}\right)(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1) - (\Delta\underline{\underline{a}}_1)_{GT} + (\Delta\underline{\underline{a}}_1)_T$$

where we deduce, using the previous expression of $(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}}$ and Eq.(41):

$$\begin{aligned}
\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}} &= \underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1 + (z + \Delta z)(\underline{\underline{a}}_1 + \Delta\underline{\underline{a}}_1 - \underline{\underline{a}}_2 - \Delta\underline{\underline{a}}_2) \\
&= \left[1 + (z + \Delta z)\frac{H}{3}(A + BY)\left(1 + \frac{F(\underline{\underline{s}} - \underline{\underline{a}}_1)}{X}\right)\right](\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}}_1 - \Delta\underline{\underline{a}}_1) \\
&+ (z + \Delta z)(\underline{\underline{a}}_1 - (\Delta\underline{\underline{a}}_1)_{GT} + (\Delta\underline{\underline{a}}_1)_T - \underline{\underline{a}}_2 - \Delta\underline{\underline{a}}_2) \\
&= \left[X + (z + \Delta z)\frac{H}{3}(A + BY)\left(X + F(\underline{\underline{s}} - \underline{\underline{a}}_1)\right)\right]\frac{\underline{\underline{s}}^*}{F(\underline{\underline{s}}^*)} \\
&+ (z + \Delta z)(\underline{\underline{a}}_1 - (\Delta\underline{\underline{a}}_1)_{GT} + (\Delta\underline{\underline{a}}_1)_T - \underline{\underline{a}}_2 - \Delta\underline{\underline{a}}_2)
\end{aligned} \tag{52}$$

(of course, it is still possible here to substitute X with $F(\underline{\underline{s}} - \underline{\underline{a}})$ in this expression).

4.2. Case where the liimit stress is reached

The discretized equations can be writeen as:

$$\Delta \underline{\underline{e}} = \Delta \underline{\underline{e}}^e + \Delta \underline{\underline{\varepsilon}}^P \tag{53}$$

$$\sigma_{eq} + \Delta\sigma_{eq} \equiv F(\underline{\underline{s}} + \Delta \underline{\underline{s}} - \underline{\underline{a}} - \Delta \underline{\underline{a}}) = \sigma^Y(\varepsilon_1^{eff} + \Delta\varepsilon_1^{eff}, \varepsilon_2^{eff} + \Delta\varepsilon_2^{eff}, z + \Delta z, T + \Delta T) \tag{54}$$

$$\Delta \underline{\underline{\sigma}}^P = \frac{3}{2} \frac{\Delta\varepsilon_{eq}}{F(\underline{\underline{s}} + \Delta \underline{\underline{s}} - \underline{\underline{a}} - \Delta \underline{\underline{a}})} (\underline{\underline{s}} + \Delta \underline{\underline{s}} - \underline{\underline{a}} - \Delta \underline{\underline{a}}) \tag{55}$$

$$\begin{aligned} &\Delta \underline{\underline{s}}_{OBJ} = \Delta \underline{\underline{s}} + (\Delta \underline{\underline{s}})_{GT} = 2\mu \Delta \underline{\underline{e}}^e + (\Delta \underline{\underline{s}})_T \\ &\Rightarrow \Delta \underline{\underline{s}} = 2\mu \Delta \underline{\underline{e}}^e - (\Delta \underline{\underline{s}})_{GT} + (\Delta \underline{\underline{s}})_T \end{aligned} \tag{56}$$

$$\Delta\varepsilon_1^{eff} = \Delta\varepsilon_{eq} \tag{57}$$

$$\begin{aligned} &\Delta(z\varepsilon_2^{eff}) = \left(z + \frac{\Delta z}{2}\right) \Delta\varepsilon_{eq} + \theta\varepsilon_1^{eff} \Delta z \Rightarrow \\ &\Delta\varepsilon_2^{eff} = \frac{1}{z + \Delta z} \left[\left(z + \frac{\Delta z}{2}\right) \Delta\varepsilon_{eq} + (-\varepsilon_2^{eff} + \theta\varepsilon_1^{eff}) \Delta z \right] \end{aligned} \tag{58}$$

$$\begin{aligned} &(\Delta \underline{\underline{a}}_1)_{OBJ} = \Delta \underline{\underline{a}}_1 + (\Delta \underline{\underline{a}}_1)_{GT} = \frac{2}{3}(1-p) \frac{\Delta\overline{\sigma}_1}{\Delta\varepsilon_1^{eff}} \Delta\varepsilon^p + (\Delta \underline{\underline{a}}_1)_T \Rightarrow \\ &\Delta \underline{\underline{a}}_1 = \frac{2}{3}(1-p) \frac{\Delta\overline{\sigma}_1}{\Delta\varepsilon_1^{eff}} \Delta\varepsilon^p - (\Delta \underline{\underline{a}}_1)_{GT} + (\Delta \underline{\underline{a}}_1)_T \end{aligned} \tag{59}$$

$$\Delta(z\underline{\underline{a}}_2) = \frac{2}{3}(1-p)(z + \frac{\Delta z}{2}) \frac{\Delta\overline{\sigma}_2}{\Delta\varepsilon_2^{eff}} \Delta\varepsilon^p + \theta \underline{\underline{a}}_1 \Delta z - z(\Delta \underline{\underline{a}}_2)_{GT} + z(\Delta \underline{\underline{a}}_2)_T \tag{60}$$

The quantities $\Delta\overline{\sigma}_1/\Delta\varepsilon_1^{eff}$ and $\Delta\overline{\sigma}_2/\Delta\varepsilon_2^{eff}$ intervening in the evolutions equations of the parameters of the kinematic hardening are here the secant of strain hardening defined by

$$\frac{\Delta\overline{\sigma}_i}{\Delta\varepsilon_i^{eff}} = \frac{1}{\Delta\varepsilon_i^{eff}} \left[\overline{\sigma}_i(\varepsilon_i^{eff} + \Delta\varepsilon_i^{eff}, T + \Delta T) - \overline{\sigma}_i(\varepsilon_i^{eff}, T + \Delta T) \right] \tag{61}$$

This choice rather than that of the slopes of work hardening, as previously, is justified by compatibility with the resolution which follows, which will naturally make use again of the secants, this time for the isotropic part of the work hardening, via the exact respect of the criterion at the time $t + \Delta t$. Note also that Eq.(60) will be used in the given form, and not in the form of an expression of $\Delta\underline{\underline{a}}_2$ which will be less convenient here.

Now let us solve these equations by adopting $\Delta\varepsilon_{eq}$ as a key unknown. Proceeding as before from Eq.(53) and Eq.(56), we obtain

$$\underline{\underline{s}} + \Delta\underline{\underline{s}} = \underline{\underline{s}} + 2\mu\Delta\underline{\underline{e}} - (\Delta\underline{\underline{s}})_{GT} + (\Delta\underline{\underline{s}})_T - 2\mu\Delta\varepsilon^p$$

which, by assuming as previously

$$(\underline{\underline{s}} + \Delta\underline{\underline{s}})^{el} = \underline{\underline{s}} + 2\mu\Delta\underline{\underline{e}} - (\Delta\underline{\underline{s}})_{GT} + (\Delta\underline{\underline{s}})_T \tag{62}$$

and using Eq.(55), is equivalent to

$$\underline{\underline{s}} + \Delta\underline{\underline{s}} = (\underline{\underline{s}} + \Delta\underline{\underline{s}})^{el} - 3\mu\frac{\Delta\varepsilon_{eq}}{F(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}})}(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}})$$

By adding $-\underline{\underline{a}} - \Delta\underline{\underline{a}}$ to the two sides of the equations, and by writing $\underline{\underline{a}} + \Delta\underline{\underline{a}}$ in the form

$$\underline{\underline{a}} + \Delta\underline{\underline{a}} = (1 - z - \Delta z)(\underline{\underline{a}}_1 + \Delta\underline{\underline{a}}_1) + z\underline{\underline{a}}_2 + \Delta(z\underline{\underline{a}}_2)$$

and using Eq.(59) and Eq.(60), we get

$$\begin{aligned}
\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}} = {} & (\underline{\underline{s}} + \Delta\underline{\underline{s}})^{el} - 3\mu\frac{\Delta\varepsilon_{eq}}{F(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}})}(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}}) \\
& - (1 - z - \Delta z)(\underline{\underline{a}}_1 + \Delta\underline{\underline{a}}_1) - z\underline{\underline{a}}_2 - \Delta(z\underline{\underline{a}}_2) \\
= {} & (\underline{\underline{s}} + \Delta\underline{\underline{s}})^{el} - (1 - z - \Delta z)\left[\underline{\underline{a}}_1 - (\Delta\underline{\underline{a}}_1)_{GT} + (\Delta\underline{\underline{a}}_1)_T\right] - z\underline{\underline{a}}_2 \\
& - 3\mu\frac{\Delta\varepsilon_{eq}}{F(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}})}(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}}) \\
& - (1 - z - \Delta z)\frac{2}{3}(1 - p)\frac{\Delta\overline{\sigma}_1}{\Delta\varepsilon_1^{eff}} \cdot \frac{3}{2}\frac{\Delta\varepsilon_{eq}}{F(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}})}(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}}) \\
& - \frac{2}{3}(1 - p)(z + \frac{\Delta z}{2})\frac{\Delta\overline{\sigma}_2}{\Delta\varepsilon_2^{eff}} \cdot \frac{3}{2}\frac{\Delta\varepsilon_{eq}}{F(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}})}(\underline{\underline{s}} + \Delta\underline{\underline{s}} - \underline{\underline{a}} - \Delta\underline{\underline{a}}) \\
& - \theta\underline{\underline{a}}_1\Delta z + z(\Delta\underline{\underline{a}}_2)_{GT} - z(\Delta\underline{\underline{a}}_2)_T
\end{aligned}$$

By using

$$\begin{aligned}
\underline{\underline{s}}^* \quad & \equiv (\underline{\underline{s}} + \Delta\underline{\underline{s}})^{el} - (1 - z - \Delta z)\left[\underline{\underline{a}}_1 - (\Delta\underline{\underline{a}}_1)_{GT} + (\Delta\underline{\underline{a}}_1)_T\right] \\
& - z\underline{\underline{a}}_2 - \theta\underline{\underline{a}}_1\Delta z + z(\Delta\underline{\underline{a}}_2)_{GT} - z(\Delta\underline{\underline{a}}_2)_T
\end{aligned} \tag{63}$$

(note that this definition is not the same as that of Eq.(37), in the case where the stress limit is not reached), and

$$\widetilde{H} \equiv (1 - z - \Delta z)(1 - p)\frac{\Delta\overline{\sigma}_1}{\Delta\varepsilon_1^{eff}} + (z + \frac{\Delta z}{2})(1 - p)\frac{\Delta\overline{\sigma}_2}{\Delta\varepsilon_2^{eff}} \tag{64}$$

this can be written as

$$\left[1+\frac{\left(\widetilde{H}+3\mu\right)\Delta\varepsilon_{eq}}{F(\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}-\Delta\underline{\underline{a}})}\right](\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}-\Delta\underline{\underline{a}})=\underline{\underline{s}}^{*} \tag{65}$$

Before going any further, let us give a simple and more meaningful expression of $\underline{\underline{s}}^{*}$. Let us denote $\underline{\underline{a}}_1+(\Delta\underline{\underline{a}}_1)_{GT,T}$ and $\underline{\underline{a}}_2+(\Delta\underline{\underline{a}}_2)_{GT,z,T}$ the values of $\underline{\underline{a}}_1$ and $\underline{\underline{a}}_2$ obtained by taking into account, in the variation $\Delta\underline{\underline{a}}_1$ and $\Delta\underline{\underline{a}}_2$, only the terms due to large transformations and variations of z and T (that is omitting the term proportional to $\Delta\underline{\underline{\varepsilon}}^p$). We have, by Eq.(59) and Eq.(60):

$$\underline{\underline{a}}_1+(\Delta\underline{\underline{a}}_1)_{GT,T}=\underline{\underline{a}}_1-(\Delta\underline{\underline{a}}_1)_{GT}+(\Delta\underline{\underline{a}}_1)_{T},$$

$$\begin{aligned}&(z+\Delta z)\lfloor\underline{\underline{a}}_2+(\Delta\underline{\underline{a}}_2)_{GT,z,T}\rfloor-z\underline{\underline{a}}_2=\theta\underline{\underline{a}}_1\Delta z-z(\Delta\underline{\underline{a}}_2)_{GT}+z(\Delta\underline{\underline{a}}_2)_T\\ &\Rightarrow(z+\Delta z)\lfloor\underline{\underline{a}}_2+(\Delta\underline{\underline{a}}_2)_{GT,z,T}\rfloor=z\underline{\underline{a}}_2+\theta\underline{\underline{a}}_1\Delta z-z(\Delta\underline{\underline{a}}_2)_{GT}+z(\Delta\underline{\underline{a}}_2)_T\end{aligned}$$

From these two expressions and Eq.(63) we can deduce that

$$\underline{\underline{s}}^{*}=(\underline{\underline{s}}+\Delta\underline{\underline{s}})^{el}-(1-z-\Delta z)\lfloor\underline{\underline{a}}_1+(\Delta\underline{\underline{a}}_1)_{GT,z,T}\rfloor-(z+\Delta z)\lfloor\underline{\underline{a}}_2+(\Delta\underline{\underline{a}}_2)_{GT,z,T}\rfloor \tag{66}$$

This expression allows an easy calculation of $\underline{\underline{s}}^{*}$, having previously carried out the pre-corrections of $\underline{\underline{a}}_1$ and $\underline{\underline{a}}_2$ due to large transformations and variations of z and T. Eq.(65) implies that the (unknown) tensor $\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}-\Delta\underline{\underline{a}}$ is positively collinear with the (known) tensor, $\underline{\underline{s}}^{*}$; thereby

$$\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}-\Delta\underline{\underline{a}}=\frac{F(\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}-\Delta\underline{\underline{a}})}{F(\underline{\underline{s}}^{*})}\underline{\underline{s}}^{*} \tag{67}$$

In addition, we obtain by taking the von Mises function of the two sides of Eq.(65):

$$F(\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}-\Delta\underline{\underline{a}})+(\widetilde{H}+3\mu)\Delta\varepsilon_{eq}=F(\underline{\underline{s}}^{*}) \tag{68}$$

The equation Eq.(54) gives, by expliciting the yield limit thanks Eq.(6):

$$\begin{aligned}F(\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}-\Delta\underline{\underline{a}})&=[1-f(z+\Delta z)]\,\sigma_1^Y(\varepsilon_1^{eff}+\Delta\varepsilon_1^{eff},T+\Delta T)\\ &\quad+f(z+\Delta z)\sigma_2^Y(\varepsilon_2^{eff}+\Delta\varepsilon_2^{eff},T+\Delta T)\\ &=[1-f(z+\Delta z)]\left[\sigma_1^Y(\varepsilon_1^{eff},T+\Delta T)+p\frac{\Delta\overline{\sigma}_1}{\Delta\varepsilon_1^{eff}}\Delta\varepsilon_1^{eff}\right]\\ &\quad+f(z+\Delta z)\left[\sigma_2^Y(\varepsilon_1^{eff},T+\Delta T)+p\frac{\Delta\overline{\sigma}_2}{\Delta\varepsilon_2^{eff}}\Delta\varepsilon_2^{eff}\right]\\ &=\sigma^Y(\varepsilon_1^{eff},\varepsilon_2^{eff},z+\Delta z,T+\Delta T)\\ &\quad+[1-f(z+\Delta z)]\,p\frac{\Delta\overline{\sigma}_1}{\Delta\varepsilon_1^{eff}}\Delta\varepsilon_1^{eff}+f(z+\Delta z)p\frac{\Delta\overline{\sigma}_2}{\Delta\varepsilon_2^{eff}}\Delta\varepsilon_2^{eff}\end{aligned}$$

which gives, by reporting in Eq.(68):

$$\sigma^Y(\varepsilon_1^{eff}, \varepsilon_2^{eff}, z+\Delta z, T+\Delta T) + [1-f(z+\Delta z)]\, p\frac{\Delta\overline{\sigma}_1}{\Delta\varepsilon_1^{eff}}\Delta\varepsilon_1^{eff}$$
$$+ f(z+\Delta z)p\frac{\Delta\overline{\sigma}_2}{\Delta\varepsilon_2^{eff}}\Delta\varepsilon_2^{eff} + (\widetilde{H}+3\mu)\Delta\varepsilon_{eq} = F(\underline{\underline{s}}^*)$$

This equation can be written as, according to Eq.(57) and Eq.(58):

$$(H+3\mu)\Delta\varepsilon_{eq} = \Delta \tag{69}$$

$$\begin{aligned} H \quad & \equiv \widetilde{H} + [1-f(z+\Delta z)]\, p\frac{\Delta\overline{\sigma}_1}{\Delta\varepsilon_1^{eff}} + f(z+\Delta z)\frac{z+\Delta z/2}{z+\Delta z}p\frac{\Delta\overline{\sigma}_2}{\Delta\varepsilon_2^{eff}} \\ & = (1-z-\Delta z)(1-p)\frac{\Delta\overline{\sigma}_1}{\Delta\varepsilon_1^{eff}} + \left(z+\frac{\Delta z}{2}\right)(1-p)\frac{\Delta\overline{\sigma}_2}{\Delta\varepsilon_2^{eff}} \\ & + [1-f(z+\Delta z)]\, p\frac{\Delta\overline{\sigma}_1}{\Delta\varepsilon_1^{eff}} + f(z+\Delta z)\frac{z+\Delta z/2}{z+\Delta z}p\frac{\Delta\overline{\sigma}_2}{\Delta\varepsilon_2^{eff}} \end{aligned} \tag{70}$$

$$\Delta \equiv F(\underline{\underline{s}}^*) - \sigma_1^Y(\varepsilon_1^{eff}+\Delta\varepsilon_1^{eff}, T+\Delta T) + f(z+\Delta z)p\frac{\Delta\overline{\sigma}_2}{\Delta\varepsilon_2^{eff}}\frac{\Delta z}{z+\Delta z}(\varepsilon_2^{eff}-\theta\varepsilon_1^{eff}) \tag{71}$$

Eq.(69) relates to the only unknown $\Delta\varepsilon_{eq}$, the strain hardening secants depends on the ε_1^{eff} which are expressed as a function of $\Delta\varepsilon_{eq}$, thanks to the equations Eq.(57) and Eq.(58). It can be solved, for example, by the method of the fixed point. The quantity $F(\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}-\Delta\underline{\underline{a}})$ is then deduced from Eq.(54), and then the tensor $\underline{\underline{s}}+\Delta\underline{\underline{s}}-\underline{\underline{a}}-\Delta\underline{\underline{a}}$ is deduced from Eq.(67). Finalyy, we calculate $\Delta\underline{\underline{\varepsilon}}^p$ thanks to Eq.(55), then $\underline{\underline{a}}_1$ and $\underline{\underline{a}}_2$ thanks to Eq.(59) and Eq.(60).

4.3. Particular case: isotropic strain hardening with the yield limit not reached

If the yield limit is not reached, the expression of Eq.(27) proposed for $(\Delta\sigma_1^{eq})_s$ is applicable whatever the type of the work hardening. However, for a pure isotropic work hardening $p \equiv 1$ is equivalent to the expression Eq.(10) of $(\dot{\sigma}_1^{eq})_s$ can also be written equivalently (with ($\underline{\underline{a}}_1 \equiv \underline{\underline{0}}$)) as:

$$(\dot{\sigma}_1^{eq})_s \equiv \frac{3}{2\sigma^{eq}}\underline{\underline{s}} : \overset{\vee}{\underline{\underline{s}}} = \dot{\sigma_{eq}} \quad , \quad \sigma_{eq} \equiv \left(\frac{3}{2}\underline{\underline{s}} : \underline{\underline{s}}\right)^{\frac{1}{2}}$$

We can then assume that a simple expression for $(\Delta\sigma_1^{eq})_s$, than Eq.(27)

$$(\Delta\sigma_1^{eq})_s \equiv Y \equiv \Delta\sigma_{eq} = F(\underline{\underline{s}} + \Delta\underline{\underline{s}}) - F(s) \equiv X - F(\underline{\underline{s}}) \tag{27''}$$

This simplification is adopted in several finite element codes. It is necessary to take again the elements of the numerisation exposed in the Section 4.1 in the case of the purely isotropic work hardening where the yield stress is not reached.

The equation Eq.(40) being obtained without using the expression Eq.(27) of $(\Delta\sigma_1^{eq})_s \equiv Y \equiv \Delta\sigma_{eq}$ is valid here also; it can be written as, with $\underline{\underline{a}}_1 \equiv \underline{\underline{0}}$, $\Delta\underline{\underline{a}}_1 \equiv \underline{\underline{0}}$, $p \equiv 1$ (thus, $H = 0$ from Eq.(38)):

$$\left[1 + \mu(A + BY)\left(1 + \frac{F(\underline{\underline{s}})}{X}\right)\right](\underline{\underline{s}} + \Delta\underline{\underline{s}}) = \underline{\underline{s}}^*$$

$\underline{\underline{s}}^*$ is always given by Eq.(37), with $\underline{\underline{a}}_1 \equiv \underline{\underline{0}}$, $(\Delta\underline{\underline{a}}_1)_T \equiv \underline{\underline{0}}$

$$X + \mu(A + BY)(X + F(\underline{\underline{s}})) = X + \mu[A + B(X - F(\underline{\underline{s}}))](X + F(\underline{\underline{s}})) = F(\underline{\underline{s}}^*)$$

The equation Eq.(41) then applied always, with $\underline{\underline{a}}_1 \equiv \underline{\underline{0}}$, $\Delta\underline{\underline{a}}_1 \equiv \underline{\underline{0}}$. In addition, taking into account the function of von Mises of the two sides of Eq.(40") and taking into account Eq.(27"), we obtain:

$$L''X^2 + M''X + N'' = 0 \tag{44''}$$

which gives by re-ordering the terms

$$L'' + \mu B \tag{45''}$$

$$M'' = 1 + \mu A \tag{46''}$$

$$N'' = \mu AF(\underline{\underline{s}}) - \mu BF^2(\underline{\underline{s}}) - F(\underline{\underline{s}}^*) \tag{47''}$$

As in the usual case, this formulation does not necessarily ensure that there exists a positive real solution X. If this is not the case, we can adopt a completely implicit algorithm (replacement of Eq.(23) and Eq.(24) by Eq.(23') and Eq.(24'). This leads to replacing $F(\underline{\underline{s}})$ by $F(\underline{\underline{s}} + \Delta\underline{\underline{s}} \equiv X$ in Eq.(40'), which becomes:

$$[1 + 2\mu(A + BY)](\underline{\underline{s}} + \Delta\underline{\underline{s}}) = \underline{\underline{s}}^* \tag{40'''}$$

By taking the von Mises function of the two sides of the previous equations and taking into account Eq.(27"), we then obtain

$$X + 2\mu X[A + B(X - F(\underline{\underline{s}}))] = F(\underline{\underline{s}}^*)$$

which is equivalent to

$$L''' X^2 + M''' X + N''' = 0 \quad (44"')$$

$$L''' + 2\mu B \quad (45"')$$

$$M''' = 1 + 2\mu(A - BF(\underline{\underline{s}})) \quad (46"')$$

$$N''' = -F(\underline{\underline{s}}^*) \quad (47"')$$

Since $L''' > 0$ and $N''' > 0$, the existence of this unique psoitve solution is therefore guaranteed.

References

[1] J.B. Leblond, G. Mottet, J.C. Devaux (1989) A theoretical and numerical approach to the plastic behaviour of steels during phase transformationsI. Derivation of general relations, Journal of the Mechanics and Physics of Solids, Volume 34, Issue 4, 1986, Pages 395-409, ISSN 0022-5096

[2] J.B. Leblond, G. Mottet, J.C. Devaux, A theoretical and numerical approach to the plastic behaviour of steels during phase transformationsII. Study of classical plasticity for ideal-plastic phases, Journal of the Mechanics and Physics of Solids, Volume 34, Issue 4, 1986, Pages 411-432, ISSN 0022-5096