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# Numerisation of a constitutive model of ductile damage in porous metals accounting for damage delocalization, cavities' nucleation, and mixed isotropic-kinematic hardening

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## 1. INTRODUCTION

The purpose of this report is to present the numerical implementation of a model of ductile rupture. This model, which was developed by G. Perrin and Leblond [8], presents a certain number of theoretical improvements compared to the model "$R/R_0$" and Rousselier model:

- better consideration of the interactions between growth of cavities and hardening, and introduction of the possibility of kinematic or mixed isotropic/kinematic hardening;
- better modeling of coalescence;
- taking into account the nucleation, brutal or continuous, of the cavities, which can allow for the simulation of the behavior of specimens in stainless steel aged by irradiation;
- incorporation of damage relocation into the model itself, not just through the imposition of a minimum mesh size. This makes it possible to overcome the usual restrictions on the shape and size of the meshes.

In addition, the experience has revealed other advantages of the new model and/or its numerical implementation, of a more computational nature:

- The new model accommodates square meshes, unlike that of Rousselier, which required the use of elongated rectangular meshes forcing the crack to remain in its plane. This made it possible to simulate the "cup-cone fracture" experiment, in which the crack deviates at $45°$ from its initial path[1]. Coalescence, neglected in the Rousselier model, is only taken into account in a relatively rough way in the $R/R'_0$ model, due to the absence of coupling plasticity damage.
- The use of an explicit algorithm with respect to the porosity ensures much better convergence than previously, which authorizes the continuation of the simulations until the complete rupture of the specimen studied [8].
- The new model adequately reproduces the behavior of homothetic test specimens of small dimensions, a result which could not be obtained until now. The precise origin of this improvement, however, is unclear.

## 2. THEORETICAL EQUATIONS of the MODEL

The model is written in large deformations, 'in Eulerian formulation, the deformation rate $\mathbf{d}$ is supposed to admit the usual additive decomposition:

$$\mathbf{d} = \mathbf{d}^{\mathrm{e}} + \mathbf{d}^{\mathrm{p}} \tag{1}$$

The model equations include the expression of the elastic strain rate $\mathbf{d}^{\mathrm{e}}$, that of the plastic strain rate $\mathbf{d}^{\mathrm{p}}$ (plasticity criterion and associated flow rule); and the expressions for the evolution equations for the internal parameters (porosity and hardening parameters).

[1] The coalescence, neglected in the model of Rousselier, is only taken into account in a relatively coarse way in the model $R/R'_0$, because of the absence of coupling plasticity damage.

*2.1. Elastic Strain Rate*

The law of elasticity used is in fact (as usual in elasto-plasticity large deformations) a law of hypo-elasticity or weak elasticity (linear relation between the rate of stress and the rate of elastic deformation; it is written, at constant temperature :

$$\hat{\boldsymbol{\Sigma}} = \lambda\,(\mathrm{tr}\mathbf{d}^{\mathrm{e}})\,1 + 2\mu\mathbf{d}^{\mathrm{e}} \tag{2}$$

where $\lambda$ and $\mu$ denote the Lamé coefficients and $\sigma$ an objective derivative of the stress tensor $\propto$. In practice, two derivatives are used: that of Jaumann, defined by:

$$\hat{\boldsymbol{\Sigma}} = \dot{\boldsymbol{\Sigma}} + \boldsymbol{\Sigma}.\boldsymbol{\Omega} - \boldsymbol{\Omega}.\boldsymbol{\Sigma} \tag{3}$$

where $\boldsymbol{\Omega} = \frac{1}{2}\left(\nabla_{\mathbf{x}}\mathbf{U} - {}^{T}\nabla_{\mathbf{x}}\mathbf{U}\right)$ ($\mathbf{x}$ current position vector, $\mathbf{U}$ velocity vector) denote the rate of rotation, and that of Green-Naghdi, defined by the same formula Eq.(3 ) , but $\boldsymbol{\Omega}$ then being equal to $\mathbf{R}.\mathbf{R}^{-1}$ where $\mathbf{R}$ is the rotation involved in the polar decomposition of the the deformation gradient.

It should be noted that this model does not incorporate a damage-elasticity coupling ( $\lambda$ and $\mu$ do not depend on the porosity and are therefore constant if the temperature is), much less important in practice than the damage-plasticity coupling. In the case where the temperature $\theta$ varies, we add to the expression of $\sigma$ a term proportional to $\dot{\theta}$ :

$$\hat{\boldsymbol{\Sigma}} = \lambda\,(\mathrm{tr}\mathbf{d}^{\mathrm{e}})\,\mathbf{I} + 2\mu\mathbf{d}^{\mathrm{e}} + \frac{\mathrm{dE}}{\mathrm{E}d\theta}\boldsymbol{\Sigma}\dot{\theta}. \tag{4}$$

where $\mathrm{E}$ denotes the Young's modulus. This formula implicitly assumes the temperature-independent Poisson's ratio. It is recalled that it ensures the cancellation of the stresses at high temperatures, and is reduced after integration with the traditional formula

$$\boldsymbol{\Sigma} = \lambda(\theta)\,(\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}})\,\mathbf{I} + 2\mu(\theta)\boldsymbol{\varepsilon}^{\mathrm{e}} \tag{5}$$

in the case where small deformation assumptions are made.

*2.2. Plastic Strain Rate*

Let us first consider the case of an isotropic work hardening. The criterion of plasticity is written as:

$$\phi(\boldsymbol{\Sigma}) = \frac{\Sigma_{\mathrm{eq}}^2}{\Sigma_1^2} + 2\mathrm{p}\,\mathrm{ch}\left(\frac{3}{2}\frac{\Sigma_{\mathrm{m}}}{\Sigma_2}\right) - 1 - \mathrm{p}^2 \leq 0. \tag{6}$$

In this expression, $\Sigma_{\mathrm{eq}}$ is the equivalent Von Mises stress ($= \left(\frac{3}{2}\mathrm{s}_{\mathrm{ij}}\mathrm{s}_{\mathrm{ij}}\right)^{\frac{1}{2}}$ , $\mathbf{s}$ denoting the deviator of the stress ), $\Sigma_{\mathrm{m}}$ the mean stress ($= (\frac{1}{3})\mathrm{tr}\,\Sigma$), $\Sigma_1$ and $\Sigma_2$ homogeneous quantities with constraints given in the expression will be specified later, $\mathrm{p}$ a parameter linked to the porosity $f$ by the formula

$$\mathrm{p} = \mathrm{q}f^*, \quad f^* = \begin{cases} f & \text{if } f \leq f_{\mathrm{c}} \\ f_{\mathrm{c}} + \delta\,(f - f_{\mathrm{c}}) & \text{if } f > f_{\mathrm{c}} \end{cases} \tag{7}$$

where $\mathrm{q}$ is the Tvergaard parameter, $f_{\mathrm{c}}$ the critical porosity at the beginning of coalescence and $\delta$ the accelerating factor of cavity growth.

The flow rule associated by normality with this criterion is written:

$$\mathbf{d}^{\mathrm{p}} = \eta \frac{\partial \phi}{\partial \boldsymbol{\Sigma}} \tag{8}$$

where $\eta$ denotes the plastic multiplier; by introducing the equivalent plastic strain rate:

$$\mathrm{d}_{\mathrm{eq}} = \left( \frac{2}{3} \delta^{\mathrm{p}}_{\mathrm{ij}} \quad \delta^{\mathrm{p}}_{\mathrm{ij}} \right)^{\frac{1}{2}} \tag{9}$$

where $\boldsymbol{\delta}^{\mathrm{P}}$ denotes the deviator of $\mathbf{d}^{\mathrm{P}}$, we can write this flow rule in the form of the expression:

$$\begin{cases} \boldsymbol{\delta}^{\mathrm{P}} = \dfrac{3}{2} \dfrac{\mathrm{d}_{\mathrm{eq}}}{\Sigma_{\mathrm{cq}}} \mathbf{s} \\ \mathrm{d}^{\mathrm{p}}_{\mathrm{m}} = \frac{\mathrm{p}}{2} \frac{\Sigma_1^2}{\Sigma_2 \Sigma_{\mathrm{eq}}} \,\mathrm{sh} \left( \frac{3}{2} \frac{\Sigma_{\mathrm{m}}}{\Sigma_2} \right) \mathrm{d}_{\mathrm{eq}} \end{cases} \tag{10}$$

where $\mathrm{d}^{\mathrm{p}}_{\mathrm{m}} = \frac{1}{3} \,\mathrm{tr}\, \mathbf{d}^{\mathrm{P}}$ is the average part of the plastic strain rate. The evolution of the porosity is given by the following equation, which results from the approximate incompressibility (i.e., neglecting the elasticity) of the metallic matrix:

$$\dot{\mathrm{f}} = 3(1 - f) \mathrm{d}^{\mathrm{p}}_{\mathrm{m}} \tag{11}$$

Finally, $\Sigma_1$ and $\Sigma_2$ are functions of the temperature $\theta$ and of two parameters of hardening noted $\varepsilon_{\mathrm{eq}}$ and $\varepsilon_m$ [2] and defined by:

$$\varepsilon_{\mathrm{eq}} = \int_0^{\mathrm{t}} \mathrm{d}_{\mathrm{eq}} \,\mathrm{dt}, \quad \varepsilon_{\mathrm{m}} = \int_0^{\mathrm{t}} \left| \mathrm{d}^{\mathrm{p}}_{\mathrm{m}} \right| \,\mathrm{dt}. \tag{12}$$

Let us now consider the case of a kinematic hardening. The expression of the criterion of plasticity is:

$$\frac{\Sigma_{\mathrm{eq}}^2}{\Sigma_0^2} + 2\mathrm{p} \,\mathrm{ch} \left( \frac{3}{2} \frac{\Sigma_{\mathrm{m}} - \alpha_{\mathrm{m}}}{\Sigma_0} \right) - 1 - \mathrm{p}^2 \leq 0 \tag{13}$$

where $\Sigma_0$ is the elastic limit (depending only on temperature) of the matrix, $\alpha_m = \frac{1}{3} \,\mathrm{tr}\, \alpha$ the mean part of the center $\alpha$ of the domain of elasticity and $\Sigma_{\mathrm{eq}}$ the equivalent Von Mises stress defined here by

$$\Sigma_{\mathrm{eq}} = \left[ \frac{3}{2} \left( s_{\mathrm{ij}} - a_{\mathrm{ij}} \right) \left( s_{\mathrm{ij}} - a_{\mathrm{ij}} \right) \right]^{\frac{1}{2}}, \tag{14}$$

$\mathbf{a}$ denoting the deviator of $\boldsymbol{\alpha}$. The expression of $\mathrm{p}$ is the same ( Eq.(**??** ) as for an isotropic hardening. The associated flow rule takes the form:

$$\begin{cases} \boldsymbol{\delta}^{\mathrm{p}} = \dfrac{3}{2} \dfrac{\mathrm{d}_{\mathrm{eq}}}{\Sigma_{\mathrm{eq}}} (\mathbf{s} - \mathbf{a}) \\ \mathrm{d}^{\mathrm{p}}_{\mathrm{m}} = \dfrac{\mathrm{p}}{2} \dfrac{\Sigma_0}{\Sigma_{\mathrm{eq}}} \,\mathrm{sh} \left( \dfrac{3}{2} \dfrac{\Sigma_{\mathrm{m}} - \alpha_{\mathrm{m}}}{\Sigma_0} \right) \mathrm{d}_{\mathrm{eq}} \end{cases} \tag{15}$$

[2]The rather complicated expressions of $\Sigma_1$ and $\Sigma_2$ as a function of $\varepsilon_{\mathrm{eq}}$ and $\varepsilon_{\mathrm{m}}$ are given in Perrin [8]

where $\delta^{\mathrm{p}}, \mathrm{d}_{\mathrm{eq}}$ and $\mathrm{d}_{\mathrm{m}}^{\mathrm{p}}$ are defined as before. The porosity evolution equation Eq.(8 is unchanged. Finally, the evolution of the center $\mathbf{a}$ of the domain of elasticity is given by:

$$\begin{cases} \hat{\mathbf{a}} = \dfrac{2}{3}\left(\dfrac{\partial\tilde{\alpha}_{\mathrm{eq}}}{\partial\varepsilon_{\mathrm{eq}}}\right)_T \boldsymbol{\delta}^{\mathrm{p}} + \dfrac{1}{\tilde{\alpha}_{\mathrm{cq}}}\left(\dfrac{\partial\tilde{\alpha}_{\mathrm{eq}}}{\partial\theta}\right)\mathbf{a}\dot{\theta} \\ \dot{\alpha}_{\mathrm{m}} = \left(\dfrac{\partial\tilde{\alpha}_{\mathrm{m}}}{\partial\varepsilon_{\mathrm{m}}}\right)_T \mathrm{d}_{\mathrm{m}}^{\mathrm{p}} + \dfrac{1}{\tilde{\alpha}_{\mathrm{m}}}\left(\dfrac{\partial\tilde{\alpha}_{\mathrm{m}}}{\partial\theta}\right)\alpha_{\mathrm{m}}\dot{\theta} \end{cases} \tag{16}$$

where $\dot{\alpha}$ denotes the same objective derivative as in Eq.(2) and Eq.(2') and where $\tilde{\alpha}_{\mathrm{eq}}$ and $\tilde{\alpha}_{\mathrm{m}}$ are functions [3] of the same hardening parameters $\varepsilon_{\mathrm{eq}}, \varepsilon_{\mathrm{m}}$ as previously. The partial derivatives $\left(\partial\tilde{\alpha}_{\mathrm{eq}}/\partial\varepsilon_{\mathrm{eq}}\right)$ and $(\partial\tilde{\alpha}_{\mathrm{m}}/\partial\varepsilon_{\mathrm{m}})$ are here taken at "triaxiality in deformation" $\mathrm{T} = \varepsilon_{\mathrm{m}}/\varepsilon_{\mathrm{eq}}$ constant. The terms proportional to $\dot{\theta}$ ensure the cancellation of $\mathbf{a}$ and $\alpha_{\mathrm{m}}$, therefore of $\boldsymbol{\Sigma}$, at high temperatures. Let us consider finally the case of a mixed isotropic/kinemtic hardening $\rho$ indicating the proportion of kinematic hardening. The criterion is written:

$$\frac{\Sigma_{\mathrm{eq}}^2}{\left[\rho\Sigma_0 + (1-\rho)\Sigma_1\right]^2} + 2\rho\,\mathrm{ch}\left(\frac{3}{2}\frac{\Sigma_{\mathrm{m}} - \rho\alpha_{\mathrm{m}}}{\rho\Sigma_0 + (1-\rho)\Sigma_2}\right) - 1 - \mathrm{p}^2 \leq 0 \tag{17}$$

where

$$\Sigma_{\mathrm{eq}} = \left[\frac{3}{2}\left(\mathrm{s}_{\mathrm{ij}} - \rho\mathrm{a}_{\mathrm{ij}}\right)\left(\mathrm{s}_{\mathrm{ij}} - \rho\mathrm{a}_{\mathrm{ij}}\right)\right]^{\frac{1}{2}}, \tag{18}$$

and the flow rule

$$\begin{cases} \boldsymbol{\delta}^{\mathrm{p}} = \dfrac{3}{2}\dfrac{\mathrm{d}_{\mathrm{eq}}}{\Sigma_{\mathrm{eq}}}(\mathbf{s} - \rho\mathbf{a}) \\ d_m^p = \dfrac{p}{2}\dfrac{\left[\rho\Sigma_0 + (1-\rho)\Sigma_1\right]^2}{\left[\rho\Sigma_0 + (1-\rho)\Sigma_2\right]\Sigma_{\mathrm{eq}}}\mathrm{sh}\left(\dfrac{3}{2}\dfrac{\Sigma_{\mathrm{m}} - \rho\alpha_{\mathrm{m}}}{\rho\Sigma_0 + (1-\rho)\Sigma_2}\right)\mathrm{d}_{\mathrm{eq}}. \end{cases} \tag{19}$$

The laws of evolution of the porosity and of the hardening parameters are the same as previously.

In fact, the cavities being very generally generated by decohesion of the metal matrix around inclusions, $f$ can not become lower than its initial value $f_0$ (if $f$ is equal to $f_0$, the cavity is closed around inclusion and the latter prevents its volume from decreasing further). Consequently, all the previous equations are valid only if $f > f_0$, or else $f = f_0$ and $\dot{f} \geq 0 \Leftrightarrow sh \geq 0$ (sh representing the hyperbolic sine of :

$$\frac{3}{2}\frac{\Sigma_{\mathrm{m}}}{\Sigma_2}, \frac{3}{2}\frac{\Sigma_{\mathrm{m}} - \alpha_{\mathrm{m}}}{\Sigma_0} \text{ or } \frac{3}{2}\frac{\Sigma_{\mathrm{m}} - \rho\alpha_{\mathrm{m}}}{\rho\Sigma_0 + (1-\rho)\Sigma_2} \tag{20}$$

depending on the type of hardening). If $f = f_0$ and $\mathrm{sh} < 0$, it is necessary to write that the behavior is that of a Von Mises material, which is in fact equivalent to setting $\mathrm{p} = 0$ in the previous equations.

### *2.3. Cavities' Nucleation*

We have until now, for simplicity, implicitly ignored the phenomena of nucleation of the cavities. Let us now examine their impact. We distinguish two types of nucleation:

[3]the expresions of which are given in [8]

- sudden nucleation, governed by a stress criterion developed at the Ecole des Mines de Paris and whose expression is:

$$\Sigma_1 + \alpha\left(\tilde{\Sigma}_{eq} - \Sigma_0\right) \leq \Sigma_c \tag{21}$$

where $\Sigma_1$ denotes the greatest principal stress of the stress tensor $\Sigma, \alpha$ a dimensionless parameter, $\Sigma_c$ a critical stress and $\tilde{\Sigma}_{eq}$ the equivalent stress defined by:

$$\tilde{\Sigma}_{eq} = \left(\frac{3}{2}\, s_{ij}s_{ij}\right)^{\frac{1}{2}} \tag{22}$$

$\left(\tilde{\Sigma}_{eq}\right)$ only coincides with $\Sigma_{eq}$ in the case of isotropic hardening). As long as the inequality is strict in Eq. (13), the behavior is that of a Von Mises material ( $\Leftrightarrow$ $p = 0$ ); when the equality is achieved, $f$ "jumps" abruptly to the value $f_0$ ( and cannot then fall below this value again).

- Continuous nucleation. We distinguish in this case two contributions, denoted $\dot{f}_c$ and $\dot{f}_g$, in the growth rate $f$ porosity. The first represents the rate of increase da at the growth of the cavities, and is given by formula Eq. (8). The second represents the rate of increase due to continuous germination, and is given by an empirical equation

$$\dot{f}_g = A d_{eq} \tag{23}$$

where A is a model parameter.

*2.4. Damage Delocalization*

For some applications (involving high stress gradients), the porosity evolution equation is "delocalized". We then define local rates of increase of porosity by growth and germination $\dot{f}_{cl}$ and $\dot{f}_{gl}$, given by formulas Eq.(8) and Eq.(15) respectively, and the true (non-local) growth rate is then given by the convolution formula:

$$\begin{aligned} f(\mathbf{x}) &= \tfrac{1}{C(\mathbf{x})} \int_\Omega \dot{f}_l(\mathbf{y})\chi(\mathbf{x}-\mathbf{y})d\Omega_y\ , \\ \dot{f}_l(\mathbf{y}) &\equiv \dot{f}_{cl}(\mathbf{y}) + \dot{f}_{gl}(\mathbf{y})\ , \\ C(\mathbf{x}) &= \int_\Omega \chi(\mathbf{x}-\mathbf{y})d\Omega_\mathbf{y}\ . \end{aligned} \tag{24}$$

$\Omega$ denotes here the studied domain and $\chi$ a weighting function, which we take Gaussian in practice:

$$\chi(\mathbf{z}) = \exp\left(\frac{-|\mathbf{z}|^2}{\mathbf{l}^2}\right), \tag{25}$$

$\mathbf{l}$ being a characteristic length (of the order of the spacing between cavities), which plays the same role as the minimum mesh size in the Rousselier model. Note that this delocalization study was thouroughly study by Enakoutsa et al. ( [1], [2], [3], [4], [5], [6] , [7])

## 3. NUMERICAL IMPLEMENTATION

Compared to the numerical implemetation of the classical elastoplasticity equations in large transformations, that of the ductile rupture models presents certain differences which mainly concern the so-called operation of "plastic stress correction" (calculation of the stresses and various other mechanical parameters at time $t + \Delta t$, knowing these quantities at time $t$ as well as the increment $\Delta\varepsilon \equiv \mathbf{d}\Delta t$ of total deformation between these two times).

*3.1. Correction of the Stress - Case of Isotropic Hardening*

Let us write the discretized equations of the problem, denoting $\Delta\boldsymbol{\varepsilon} \equiv \mathbf{d}\Delta t$, $\Delta\boldsymbol{\varepsilon}^e \equiv \mathbf{d}^e\Delta t$, $\Delta\boldsymbol{\varepsilon}^p \equiv \mathbf{d}^p\Delta t$, $\Delta\mathbf{e}^e$ and $\Delta\mathbf{e}^p$ the deviatoric parts of $\Delta\boldsymbol{\varepsilon}^e$ and $\Delta\boldsymbol{\varepsilon}^p$, $\Delta\varepsilon_m^e$ and $\Delta\varepsilon_m^p$ their middle parts, assigning a ' the quantities taken at time $t + \Delta t$ (the non-primed quantities are taken at time t ), and initially neglecting the effects due temperature variations and major transformations:

- Decomposition of the deformation increment:

$$\Delta\boldsymbol{\varepsilon} = \Delta\boldsymbol{\varepsilon}^e + \Delta\boldsymbol{\varepsilon}^p \tag{26}$$

- Elasticity law :

$$\Delta\mathbf{s} = 2\mu\Delta\mathbf{e}^e, \tag{27}$$

$$\Delta\Sigma_m = (3\lambda + 2\mu)\Delta\varepsilon_m^e \tag{28}$$

- Yield criteria :

$$\frac{\Sigma_{eq}'^2}{\Sigma_1'^2} + 2\tilde{p}'\mathrm{ch}\left(\frac{3}{2}\frac{\Sigma_m'}{\Sigma_2'}\right) - 1 - \tilde{p}'^2 = 0 \tag{29}$$

- Plastic flow rule :

$$\Delta\mathbf{e}^p = \frac{3}{2}\frac{\Delta\varepsilon_{eq}}{\Sigma_{eq}'}\mathbf{s}', \tag{30}$$

$$\Delta\varepsilon_m^p = \frac{\tilde{p}''}{2}\frac{\Sigma_1'^2}{\Sigma_2'\Sigma_{eq}'}\,\mathrm{sh}\left(\frac{3}{2}\frac{\Sigma_m'}{\Sigma_2'}\right)\Delta\varepsilon_{eq}. \tag{31}$$

- Definition of $\Sigma_1'$ et $\Sigma_2'$ :

$$\begin{aligned} \Sigma_1' &\equiv \Sigma_1\left(\varepsilon_{eq}', \varepsilon_m'\right), \\ \Sigma_2' &\equiv \Sigma_2\left(\varepsilon_{eq}', \varepsilon_m'\right) \end{aligned} \tag{32}$$

- Evolution equation of the hardening parameter:

$$\begin{aligned} \Delta\varepsilon_{eq} &= \left(\tfrac{2}{3}\Delta e_{ij}^p\Delta e_{ij}^P\right)^{\frac{1}{2}}. \\ \Delta\varepsilon_m &= \left|\Delta\varepsilon_m^p\right|. \end{aligned} \tag{33}$$

Only equations Eq.(20) and Eq.(21) call for specific comments here:

- To be perfectly logical, it would be necessary to use in Eq.(20), which constitutes the writing of the criterion at the instant $t + \Delta t$, the quantity $p' \equiv p(t + \Delta t)$ deriving, via Eq.(5), from the real porosity $f'$ at this instant. This porosity being unknown, the algorithm is then implicit in relation to this variable (as in relation to the others). The numerical experiment however showed that the convergence is very difficult, even impossible, with such an algorithm, and that one can, in practice, obtain results only with an explicit algorithm compared to the porosity (but however implicit against all other parameters). We therefore replace in Eq.(20) the quantity $p'$ by an approximation noted $\tilde{p}'$ , deriving via Eq.(5), from the estimation of $\tilde{f}'$ of $f' \equiv f(t + \Delta t)$ given by

$$\tilde{f}' \equiv f(t) + \dot{f}(t)\Delta t. \tag{34}$$

This of course requires storing the rate of increase $\dot{f}$ of the porosity.

- In the e Eq.(32), which gives the increase in average plastic deformation between the times $t$ and $t + \Delta t$, the most precise would be to use the quantity $p'' = p(t + \Delta t/2)$ (deriving from the true porosity $(f''$ to $t + \Delta t/2)$ In order to preserve however the explicit character of the algorithm with respect to porosity, pn replaces $p''$ by the approximation $\tilde{p}''$ derived from the approximate porosity

$$\tilde{f}'' = f(t) + f(t)\frac{\Delta t}{2}. \tag{35}$$

The beginning of the resolution of these equations follows the classic approach: we add $\mathbf{s}$ to the two members of Eq.(19) taking into account Eq.(18) Eq.(21) :

$$\mathbf{s}' \equiv \mathbf{s} + \Delta\mathbf{s} = \mathbf{s} + 2\mu\Delta\mathbf{e} - 2\mu\Delta\mathbf{e}^p = \mathbf{s}^* - 3\mu\frac{\Delta\varepsilon_{eq}}{\Sigma'_{eq}}\mathbf{s}' \tag{36}$$

where

$$s^* \equiv \mathbf{s} + 2\mu\Delta\mathbf{e} \quad (\Delta\mathbf{e} \equiv \text{ deviator of } \Delta\boldsymbol{\varepsilon}) \tag{37}$$

is the final stress deviator “computed elastically”, i.e. assuming the increment $\Delta\varepsilon$ of purely elastic total deformation (known quantity since we know $\mathbf{s}$ and $\Delta e$ ). This implies that, as usual, $\mathbf{s}'$ and $\mathbf{s}^*$ are parallel, so that we can replace
$\mathbf{s}'/\Sigma'_{eq}$ by $\mathbf{s}^*/\Sigma^*_{eq}$ ( where $\Sigma^*_{eq} \equiv \left(\frac{3}{2}s^*{}_{ij}s^*{}_{ij}\right)^{\frac{1}{2}}\Big)$ in the flow law Eq.(32) and the calculation of $\Delta ep$ is reduced to that of $\Delta\varepsilon_{eq}$. Moreover, taking the Von Mises function of the two members, we also deduce that

$$\Sigma^*_{eq} - \Sigma'_{eq} = 3\mu\Delta\varepsilon_{eq} \tag{38}$$

again a classic equation. By adding in the same way $\Sigma_m$ to the two members of Eq.(19) taking into account Eq.(18), we obtain the same

$$\Sigma^*_m - \Sigma'_m = (3\lambda + 2\mu)\Delta\varepsilon^p_m, \tag{39}$$

where

$$\Sigma^*_m \equiv \Sigma_m + (3\lambda + 2\mu)\Delta E_m \tag{40}$$

($\Delta E_m$ mean part of $\Delta\varepsilon$)([4] denotes the ”elastically calculated” (known) final mean stress. Combining Eq. (27) and Eq. (29), we get:

$$\frac{\Delta\varepsilon^p_m}{\Delta\varepsilon_{eq}} = \frac{3\mu}{3\lambda + 2\mu}\frac{\Sigma^*m - \Sigma'_m}{\Sigma^*_{eq} - \Sigma'_{eq}}. \tag{41}$$

[4]The notation $E_m$ is used here to avoid confusion with the hardening parameter $\varepsilon_m$ .

Relating this equation to Eq. (34), we get

$$\frac{3\mu}{3\lambda+2\mu}\frac{\Sigma_m^* - \Sigma_m'}{\Sigma_{eq}^* - \Sigma_{eq}'} = \frac{\tilde{p}''}{2}\frac{\Sigma_1'^2}{\Sigma_2'\Sigma_{eq}'}\,\mathrm{sh}\left(\frac{3}{2}\frac{\Sigma_m'}{\Sigma_2'}\right). \tag{42}$$

The problem is reduced to the resolution of equations Eq. (20), Eq. (22), Eq. (22), Eq. (27), Eq. (28) and Eq. (30) with respect to the unknowns $\Sigma_{eq}', \sigma_m', \sigma_1', \Sigma_2', \Delta\varepsilon_{eq}, \Delta\varepsilon_m^p$ and $\Delta\varepsilon_m$. For this, we adopt an iterative approach with respect to the unknowns $\Delta\varepsilon_{eq}, \Delta\varepsilon_m^p, \Delta\varepsilon_m, \Sigma_1', \Sigma_2'$ : starting from certain initial values of these parameters, we solve (we will see how) Eq.(20) and Eq.(44) by compared to $\Sigma_{eq}'$ and $\Sigma_m'$, we deduce $\Delta\varepsilon_{eq}, \Delta\varepsilon_m^p$ and $\Delta\varepsilon_m$ with Eq. (27 ), Eq. (28) and Eq. (23), then $\Sigma_1'$ and $\Sigma_2'$ by Eq. (22) and we iterate the process until convergence.

The whole problem therefore consists in simultaneously solving equations Eq. (20) and Eq. (30) with respect to $\sigma_{eq}'$ and $\sigma_m'$, the other parameters being assumed to be known. For this, we use the following parametrization (inspired by that of an ellipse) of the flow surface Eq. (20):

$$\begin{cases} \Sigma_{eq}' = (1-\tilde{p}')\,\Sigma_1'\cos\varphi \\ \Sigma_m' = \frac{2}{3}\Sigma_2'\,\mathrm{sgn}(\varphi)\mathrm{Arg\,ch}\left[1+\frac{(1-\tilde{p}')^2}{2p'}\sin^2\varphi\right]. \end{cases} \quad \left(-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\right). \tag{43}$$

The problem is then to solve the following equation, taken from Eq. (30), with respect to the unique variable $\varphi$:

$$\mathrm{F}(\varphi) = \mathrm{a}\left[\sigma_m^* - \Sigma_m'(\varphi)\right]\cos\varphi\cdot\tilde{p}''\left[\Sigma_{eq}^* - \Sigma_{eq}'(\varphi)\right]\mathrm{sh}\left[\frac{3}{2}\frac{\Sigma_m'(\varphi)}{\Sigma_2'}\right] = 0 \tag{44}$$

where

$$\mathrm{a} \equiv 2\,(1-\tilde{p}')\cdot\frac{\Sigma_2'}{\Sigma_1'}\cdot\frac{3\mu}{3\lambda+2\mu} \tag{45}$$

It is enough for that to use the method of Newton; one easily calculates for this purpose:

$$\begin{aligned} F(\varphi) = &-\sin\varphi\left(\mathrm{a}\left[\sigma_m^* - \Sigma_m'(\varphi) + \frac{2\,(1-\tilde{p}')^2\,\Sigma_2'\cos^2\varphi}{3\tilde{p}'\,\mathrm{sh}\left(\frac{3}{2}\frac{\Sigma_m'(\varphi)}{\Sigma_2'}\right)}\right]\right. \\ &+\tilde{p}''\,(1-\tilde{p}')\,\sigma_1'\mathrm{sh}\left(\frac{3}{2}\frac{\sigma_m'(\varphi)}{\Sigma_2'}\right) \\ &\left.+\frac{\tilde{p}''}{\tilde{p}'}\,(1-\tilde{p}')^2\left[\Sigma_{eq}^* - \Sigma_{eq}'(\varphi)\right]\cos\varphi\,\mathrm{coth}\left(\frac{3}{2}\frac{\Sigma_m'(\varphi)}{\Sigma_2'}\right)\right\}. \end{aligned} \tag{46}$$

Most of the numerical solution therefore consists of two nested loops, the outer loop carrying out the iterations on the parameters $\sigma_1', \sigma_2', \Delta\varepsilon_{eq}, \Delta\varepsilon_m^p$ and $\Delta\varepsilon_m$, the inner loop solving Eq.(46) by the Newton's method. Once this calculation is complete, the program evaluates the local porosity increment $\Delta f_1$ using the following discretized version of equations Eq. (8) and Eq. (15):

$$\Delta f_1 = \Delta f_{lc} + \Delta f_{lg}, \Delta f_{lc} = 3\left(1-\tilde{f}'\right)\Delta\varepsilon_m^p, \Delta f_{lg} = \mathrm{A}\varepsilon_{eq}.$$

Let us now indicate the modifications made by taking into account temperature variations and major transformations. It is then necessary to add the increment of thermal deformation $\Delta\varepsilon^t$ in the second member of Eq.

(18). In addition, equations Eq. (19) should be replaced by:

$$\begin{cases} \Delta\mathbf{s} + (\Delta\mathbf{s})_{\mathrm{J\ or\ M}} \equiv 2\mu'\Delta\mathbf{e}^{\mathrm{e}} + \frac{\Delta \mathrm{E}}{\mathrm{E}}\mathbf{s}, (\Delta\mathbf{s})_{\mathrm{J\ or\ M}} \equiv \Sigma.\Delta\boldsymbol{\Omega} - \Delta\boldsymbol{\Omega}.\Sigma \\ \Delta\Sigma_{\mathrm{m}} = (3\lambda' + 2\mu')\,\Delta\varepsilon^{\mathrm{e}}_{\mathrm{m}} + \frac{\Delta \mathrm{E}}{\mathrm{E}}\Sigma_{\mathrm{m}} \end{cases} \tag{47}$$

where $\Delta\boldsymbol{\Omega} = \boldsymbol{\Omega}\Delta\mathrm{t}$ represents the rotation increment[5] . In these equations, the Lamé coefficients $\lambda'$ and $\mu'$ are taken at time $\mathrm{t} + \Delta\mathrm{t}$, but the Young's modulus E and the stresses $\boldsymbol{\Sigma}, \mathbf{s}, \Sigma_{\mathrm{m}}$ at the time t.

The approach is then the same as before, but adding $\mathbf{s} - (\Delta\mathbf{s})_{\mathrm{J}\ or\ \mathrm{M}}$ to the two members of Eq. (19) instead of $\mathbf{s}$; the equations obtained are the same as before provided that expressions Eq.(26) and Eq.(29) of $\mathbf{s}^*$ and $\Sigma^*_{\mathrm{m}}$ are modified as follows:

$$\begin{aligned} &\mathbf{s}^* = \mathbf{s} + 2\mu'\Delta\mathbf{e}^{\mathrm{e}} - (\Delta\mathbf{s})_{\mathrm{J\ or\ M}} + \tfrac{\Delta \mathrm{E}}{\mathrm{E}}\mathbf{s}, \\ &\Sigma^*_{\mathrm{m}} = \Sigma_{\mathrm{m}} + (3\lambda' + 2\mu')\,(\Delta E_{\mathrm{m}} - \Delta\varepsilon^{\mathrm{t}}_{\mathrm{m}}) + \tfrac{\Delta \mathrm{E}}{\mathrm{E}}\Sigma_{\mathrm{m}}. \end{aligned} \tag{48}$$

The rest of the resolution is unchanged except for these changes (and the $\lambda \to \lambda'$ and $\mu \to \mu'$ substitutions). In practice, the corrective terms $-(\Delta\mathbf{s})_{\mathrm{J\ or\ M}}$, $\frac{\Delta \mathrm{E}}{\mathrm{E}}\mathbf{s}, -(3\lambda' + 2\mu')\,\Delta\varepsilon^{\mathrm{t}}$ and $\frac{\Delta \mathrm{E}}{\mathrm{E}}\Sigma_{\mathrm{m}}$ are added to $\mathbf{s}^*$ and $\Sigma^*_{\mathrm{m}}$.

### *3.2. Stress Correction - Case of Kinematics Hardening*

The discretized equations of the problem are written here, with notations analogous to those of the isotropic case:

$$\begin{aligned}
&\Delta\boldsymbol{\varepsilon} = \Delta\boldsymbol{\varepsilon}^{\mathrm{e}} + \Delta\boldsymbol{\varepsilon}^{\mathrm{p}} + \boldsymbol{\varepsilon}^{\mathrm{t}} \\
&\Delta\mathbf{s} + (\Delta\mathbf{s})_{\mathrm{J\ or\ M}} = 2\mu'\Delta\mathbf{e}^{\mathrm{e}} + \frac{\Delta \mathrm{E}}{\mathrm{E}}\mathbf{s}, \\
&\Delta\Sigma_{\mathrm{m}} = \Sigma_{\mathrm{m}} + (3\lambda' + 2\mu')\,\Delta\varepsilon^{\mathrm{e}}_{\mathrm{m}} + \frac{\Delta \mathrm{E}}{\mathrm{E}}\Sigma_{\mathrm{m}} \\
&\begin{cases} \dfrac{\Sigma'^2_{\mathrm{cq}}}{\Sigma'^2_0} + 2\tilde{\mathrm{p}}'\mathrm{ch}\left(\dfrac{3}{2}\dfrac{\Sigma'_{\mathrm{m}} - \alpha'\mathrm{m}}{\Sigma'_0}\right) - 1 - \tilde{\mathrm{p}}'^2 = 0 \\ \Sigma'_{\mathrm{eq}} = \left[\frac{3}{2}\left(s'_{\mathrm{ij}} - \mathrm{a}'_{\mathrm{ij}}\right)\left(\mathrm{s}'_{\mathrm{ij}} - \mathrm{a}'_{\mathrm{ij}}\right)\right]^{\frac{1}{2}} \end{cases} \\
&\begin{cases} \Delta\mathbf{e}^{\mathrm{p}} = \dfrac{3}{2}\dfrac{\Delta\varepsilon_{\mathrm{eq}}}{\sigma'_{\mathrm{eq}}}\left(\mathbf{s}' - \mathbf{a}'\right)\ , \\ \Delta\mathrm{e}^{\mathrm{p}}_{\mathrm{m}} = \dfrac{\tilde{\mathrm{p}}''}{2}\dfrac{\sigma'_0}{\sigma'_{\mathrm{eq}}}\,\mathrm{sh}\left(\dfrac{3}{2}\dfrac{\sigma'_{\mathrm{m}} - \alpha'_{\mathrm{m}}}{\sigma'_0}\right)\Delta\varepsilon_{\mathrm{eq}}\ . \end{cases} \\
&\begin{cases} \Delta\mathbf{a} + (\Delta\mathbf{a})_{\mathrm{Jor\ M}} = \dfrac{2}{3}\left(\dfrac{\Delta\tilde{\alpha}_{\mathrm{eq}}}{\Delta\varepsilon_{\mathrm{eq}}}\right)_T\Delta\mathbf{e}^{\mathrm{P}} + \dfrac{1}{\tilde{\alpha}_{\mathrm{eq}}}\left(\dfrac{\Delta\tilde{\alpha}_{\mathrm{eq}}}{\Delta\theta}\right)\mathbf{a}\Delta\theta, \\ (\Delta\mathbf{a})_{\mathrm{J\ or\ M}} = \mathbf{a}\cdot\Delta\Omega - \Delta\Omega.\mathbf{a} \\ \Delta\alpha_{\mathrm{m}} = \left(\dfrac{\Delta\tilde{\alpha}_{\mathrm{m}}}{\Delta\varepsilon_{\mathrm{m}}}\right)_{\mathrm{T}}\Delta\varepsilon^{\mathrm{p}}_{\mathrm{m}} + \dfrac{1}{\tilde{\alpha}_{\mathrm{m}}}\left(\dfrac{\Delta\tilde{\alpha}_{\mathrm{m}}}{\Delta\theta}\right)\alpha_{\mathrm{m}}\Delta\theta \end{cases} \\
&\tilde{\alpha}_{\mathrm{eq}} \equiv \tilde{\alpha}_{\mathrm{eq}}\left(\varepsilon_{\mathrm{eq}}, \varepsilon_{\mathrm{m}}, \theta\right), \quad \tilde{\alpha}_{\mathrm{m}} \equiv \tilde{\alpha}_{\mathrm{m}}\left(\varepsilon_{\mathrm{eq}}, \varepsilon_{\mathrm{m}}, \theta\right) \\
&\Delta\varepsilon_{\mathrm{eq}} = \left(\frac{2}{3}\Delta\mathrm{e}^{\mathrm{p}}_{\mathrm{ij}}\cdot\Delta\mathrm{e}^{\mathrm{p}}_{\mathrm{ij}}\right)^{\frac{1}{2}}, \Delta\varepsilon_{\mathrm{m}} = \left|\Delta\varepsilon^{\mathrm{p}}_{\mathrm{m}}\right|
\end{aligned} \tag{49}$$

[5]Note that the expression of $\Delta\sigma_{\mathrm{m}}$ does not include a corrective term da to the objective derivative; this is because the trace $\boldsymbol{\Sigma}.\boldsymbol{\Omega} - \boldsymbol{\Omega}.\boldsymbol{\Sigma}$ is zero (consequence of = $\mathbf{tr}(\mathbf{A}.\mathbf{B}) = \mathbf{tr}(\mathbf{B}.\mathbf{A})$)

As usual, the secants $\left(\Delta\tilde{\alpha}_{eq}/\Delta\varepsilon_{eq}\right)_T$ and $(\Delta\tilde{\alpha}_m/\Delta\varepsilon_m)_T$ are taken has the final temperature $\theta'$, and the secants $\left(\Delta\tilde{\alpha}_{eq}/\Delta\theta\right)$ and $(\Delta\tilde{\alpha}_m/\Delta\theta)$ has the initial $\left(\varepsilon_{eq},\varepsilon_m\right)$ deformation. Moreover, the first two secants are taken at constant triaxiality equal to the initial triaxiality $\mathrm{T} = \varepsilon_m/\varepsilon_{eq}$.

Adding $\mathbf{s} - (\Delta\mathbf{s})_{\mathrm{J\ or\ M}} - \mathbf{a} - \Delta\mathbf{a}$ to both members from Eq.(47) we get

$\mathbf{s}' - \mathbf{a}' \equiv \mathbf{s} + \Delta\mathbf{s} - \mathbf{a} - \Delta\mathbf{a} = \mathbf{s} + 2\mu'\Delta\mathbf{e}^{\mathrm{e}} - (\Delta\mathbf{s}))_{\mathrm{J\ or\ M}} + \frac{\Delta\mathrm{E}}{\mathrm{E}}\mathbf{s} - \mathbf{a} - \Delta\mathbf{a}$

which yields, taking into account Eq.(47), Eq.$(49)_6$, Eq.$(49)_{12}$:

$$\mathbf{s}' - \mathbf{a}' = \mathbf{s} + 2\mu'\Delta\mathbf{e} - (\Delta\mathbf{s})_{\mathrm{J\ or\ M}} + \frac{\Delta\mathrm{E}}{\mathrm{E}}\mathbf{s} - \mathbf{a} - \frac{2}{3}\left(\frac{\Delta\tilde{\alpha}_{eq}}{\Delta\varepsilon_{eq}}\right)_{\mathrm{T}} \cdot \frac{3}{2}\frac{\Delta\varepsilon_{eq}}{\sigma'_{eq}}\left(\mathbf{s}' - \mathbf{a}'\right) \tag{50}$$

$$+(\Delta\mathbf{a})_{\mathrm{J\ or\ M}} - \frac{1}{\tilde{\alpha}_{eq}}\left(\frac{\Delta\tilde{\alpha}_{eq}}{\Delta\theta}\right)\mathbf{a}\Delta\theta - 2\mu'\cdot\frac{3}{2}\frac{\Delta\varepsilon_{eq}}{\sigma'_{eq}}\left(\mathbf{s}' - \mathbf{a}'\right). \tag{51}$$

Assuming

$$\mathbf{s}^* = \mathbf{s} + 2\mu'\Delta\mathbf{e} - (\Delta\mathbf{s})_{\mathrm{J\ or\ M}} + \frac{\Delta\mathrm{E}}{\mathrm{E}}\mathbf{s} - \mathbf{a} + (\Delta\mathbf{a})_{\mathrm{J\ or\ M}} - \frac{1}{\tilde{\alpha}_{eq}}\left(\frac{\Delta\tilde{\alpha}_{eq}}{\Delta\theta}\right)\mathbf{a}\Delta\theta \tag{52}$$

($\mathbf{s}^*$ is a known quantity), this is written

$$\mathbf{s}' - \mathbf{a}' = \mathbf{s}^* - \left[3\mu' + \left(\frac{\Delta\tilde{\alpha}_{eq}}{\Delta e_{eq}}\right)_T\right]\frac{\Delta\varepsilon_{eq}}{\sigma'_{eq}}\left(\mathbf{s}' - \mathbf{a}'\right) \tag{53}$$

equation which shows that $\mathbf{s}' - \mathbf{a}'$ and $\mathbf{s}^*$ are parallel and reduces, as in the isotropic case, the computation from $\Delta\mathrm{ep}$ to that of $\Delta\varepsilon_{eq}$. Moreover, taking the Von Mises function of the two members, we obtain by setting

$$\Sigma^*_{eq} = \left(\frac{3}{2}\mathrm{s}^*\mathrm{ij}\mathrm{s}^*\mathrm{ij}\right)^{\frac{1}{2}}. \tag{54}$$

the equation

$$\Sigma^*_{eq} - \Sigma'_{eq} = \left[3\mu' + \left(\frac{\Delta\tilde{\alpha}_{eq}}{\Delta\varepsilon_{eq}}\right)_T\right]\Delta\varepsilon_{eq}, \tag{55}$$

analogous to Eq. (27) of the isotropic case.

Similarly, adding $\Sigma_m - \alpha_m - \Delta\alpha_m$ to both sides of Eq.(29) , we obtain :

$$\Sigma'_m - \alpha'_m \equiv \Sigma_m + \Delta\Sigma_m - \alpha_m - \Delta\alpha_m = \Sigma_m + (3\lambda' + 2\mu')\Delta\varepsilon^{\mathrm{e}}_m + \frac{\Delta\mathrm{E}}{\mathrm{E}}\Sigma_m - \alpha_m - \Delta\alpha_m, \tag{56}$$

which yields, taking into account Eq.$(5)_1$ and Eq.$(49)_{10}$:

$$\sigma^*_m - (\Sigma'_m - \alpha'_m) = \left[3\lambda' + 2\mu' + \left(\frac{\Delta\tilde{\alpha}_m}{\Delta\varepsilon_m}\right)_T\right]\Delta\varepsilon^{\mathrm{p}}_m, \tag{57}$$

where $\Sigma_m^*$ denotes the (known) quantity defined by :

$$\Sigma_m^* = \Sigma_m + (3\lambda' + 2\mu')\left(\Delta E_m - \Delta\varepsilon_m^t\right) + \frac{\Delta E}{E}\Sigma_m - \alpha_m - \frac{1}{\tilde{\alpha}_m}\left(\frac{\Delta\tilde{\alpha}_m}{\Delta\theta}\right)\alpha_m\Delta\theta; \tag{58}$$

Eq. (48) is analogous to Eq. $(52)_6$ in the isotropic case. Now combining Eq.$(49)_7$, Eq.(52) and Eq.(56), we get :

$$\frac{3\mu' + \left(\frac{\Delta\tilde{\alpha}_{eq}}{\Delta\varepsilon_{eq}}\right)_T}{3\lambda' + 2\mu' + \left(\frac{\Delta\tilde{\alpha}_m}{\Delta\varepsilon_m}\right)_T} \cdot \frac{\Sigma_m^* - (\Sigma'_m - \alpha'_m)}{\Sigma_{eq}^* - \Sigma'_{eq}} = \frac{\tilde{p}''}{2}\frac{\Sigma'_0}{\Sigma'_{eq}}\mathrm{sh}\left(\frac{3}{2}\frac{\Sigma'_m - \alpha'_m}{\Sigma'_0}\right), \tag{59}$$

equation analogous to Eq. (30).

From there, we adopt an iterative resolution method, as in the case of isotropic work hardening starting from initial values of the parameters

$$\Delta\varepsilon_{eq}, \Delta\varepsilon_m^p, \Delta\varepsilon_m, \left(\frac{\Delta\tilde{\alpha}_{eq}}{\Delta\varepsilon_{eq}}\right)_T, \left(\frac{\Delta\tilde{\alpha}_m}{\Delta\varepsilon_m}\right)_T, \tag{60}$$

we start by solving the equations Eq. $(30)_1$ and Eq. (30) with respect to $\Sigma_{eq}$ and $\Sigma'_m - \alpha'_m$; as these equations are identical to those Eq. (20) and Eq. (30) of the isotropic case on condition of replacing $\Sigma'_1, \Sigma'_2, 3\mu, 3\lambda+2\mu, \Sigma'_m$ with

$$\Sigma'_0, \Sigma'_0, 3\mu' + \left(\frac{\Delta\tilde{\alpha}_{eq}}{\Delta\varepsilon_{eq}}\right)_T, 3\lambda + 2\mu + \left(\frac{\Delta\vec{\alpha}_m}{\Delta\varepsilon_m}\right)_T, \Sigma'_m - \alpha'_m, \tag{61}$$

it suffices to employ the same method with these substituteior:s; then we draw $\Delta\varepsilon_{eq}, \Delta\varepsilon_m^p$ and $\Delta\varepsilon_m$ from Eq. (27), Eq. (28), Eq. $(23)_2$, we deduce

$$\left(\frac{\Delta\tilde{\alpha}_{eq}}{\Delta\varepsilon_{eq}}\right)_T \text{ et } \left(\frac{\Delta\tilde{\alpha}_m}{\Delta\varepsilon_m}\right)_T \tag{62}$$

thanks to Eq. (38) and we iterate the process until convergence.

When this calculation is finished, it is not only necessary to calculate, as in the isotropic case, the local increment of porosity $\Delta f_1$ (equationa Eq. (36)), but also to evolve $\mathbf{a}$ and $\alpha_m$ according to formula Eq. (37).

*3.3. Stress Correction - Case of Mixed Isotropic/kinematic Hardening*

The discretized equations of the problem are the same as in the kinematic case, with the exception of Eq. $(49)_5$ and Eq. $(49)_7$ which are written here:

$$\begin{cases} \dfrac{\Sigma'^2_{eq}}{\left[\rho\Sigma'_0 + (1-\rho)\Sigma'_1\right]^2} + 2\tilde{p}'\mathrm{ch}\left(\dfrac{3}{2}\dfrac{\Sigma'_m - \rho\alpha'_m}{\rho\left(\Sigma'_0 + (1-\rho)\Sigma'_2\right)}\right) - 1 - \tilde{p}^2 = 0, \\ \Sigma'_{eq} \equiv \left[\dfrac{3}{2}\left(s'_{ij} - \rho a'_{ij}\right)\left(s'_{ij} - \rho a'_{ij}\right)\right]^{\frac{1}{2}}. \end{cases} \tag{63}$$

$$\begin{cases} \Delta\mathbf{e}^p = \dfrac{3}{2}\dfrac{\Delta\varepsilon_{eq}}{\sigma'_{eq}}\left(\mathbf{s}' - \rho\mathbf{a}'\right), \\ \Delta\varepsilon_m^p = \dfrac{\tilde{p}''}{2}\dfrac{\left[\rho\sigma'_0 + (1-\rho)\sigma'_1\right]^2}{\left[\rho\sigma'_0 + (1-\rho)\sigma'_2\right]\sigma'_{eq}}\mathrm{sh}\left(\dfrac{3}{2}\dfrac{\sigma'_m - \rho\alpha'_m}{\left[\rho\sigma'_0 + (1-\rho)\sigma'_2\right]}\right)\Delta\varepsilon_{eq} \end{cases} \tag{64}$$

$$\begin{cases} \Delta\mathbf{e}^{\mathrm{p}} = \dfrac{3}{2}\dfrac{\Delta\varepsilon_{\mathrm{eq}}}{\sigma'_{\mathrm{eq}}}\left(\mathbf{s}' - \rho\mathbf{a}'\right), \\ \Delta\varepsilon^{\mathrm{p}}_{\mathrm{m}} = \dfrac{\tilde{\mathrm{p}}''}{2}\dfrac{\left[\rho\sigma'_0 + (1-\rho)\sigma'_1\right]^2}{\left[\rho\sigma'_0 + (1-\rho)\sigma'_2\right]\sigma'_{\mathrm{eq}}}\,\mathrm{sh}\left(\dfrac{3}{2}\dfrac{\sigma'_{\mathrm{m}} - \rho\alpha'_{\mathrm{m}}}{\left[\rho\sigma'_0 + (1-\rho)\sigma'_2\right]}\right)\Delta\varepsilon_{\mathrm{eq}} \end{cases} \tag{65}$$

In these equations, $\Sigma'_1$ and $\Sigma'_2$ are given by:

$\Sigma'_1 \equiv \Sigma_1(\varepsilon'_{\mathrm{eq}}, \varepsilon'_{\mathrm{m}}, \theta')$, $\Sigma'_2 \equiv \Sigma_2(\varepsilon'_{\mathrm{eq}}, \varepsilon'_{\mathrm{m}}, \theta')$.

We do not repeat here the whole approach and we will content ourselves with indicating how the final equations must be modified with respect to the kinematic case: Eq.(50) and Eq.(57) become

$$\mathbf{s}^* = \mathbf{s} + 2\mu'\Delta\mathbf{e} - (\Delta\mathbf{s})_{\mathrm{J\ or\ M}} + \frac{\Delta\mathrm{E}}{\mathrm{E}}\mathbf{s} - \rho\mathbf{a} + \rho(\Delta\mathrm{a})_{\mathrm{J\ or\ M}} - \frac{1}{\tilde{\alpha}_{\mathrm{eq}}}\left(\frac{\Delta\tilde{\alpha}_{\mathrm{eq}}}{\Delta\theta}\right)\rho\mathbf{a}\Delta\theta \tag{66}$$

$$\Sigma^*_{\mathrm{eq}} - \Sigma'_{\mathrm{eq}} = \left[3\mu' + \rho\left(\frac{\Delta\tilde{\alpha}_{\mathrm{eq}}}{\Delta\varepsilon_{\mathrm{eq}}}\right)_T\right]\Delta\varepsilon_{\mathrm{eq}} \tag{67}$$

and the equations Eq.(52) and Eq.(53)

$$\Sigma^*_{\mathrm{m}} - (\Sigma'_{\mathrm{m}} - \rho\alpha'_{\mathrm{m}}) = \left[3\lambda' + 2\mu' + \rho\left(\frac{\Delta\tilde{\alpha}_{\mathrm{m}}}{\Delta\varepsilon_{\mathrm{m}}}\right)_T\right]\Delta\varepsilon^{\mathrm{p}}_{\mathrm{m}} \tag{68}$$

$$\Sigma^*_{\mathrm{m}} = \Sigma_{\mathrm{m}} + (3\lambda' + 2\mu')\left(\Delta E_{\mathrm{m}} - \Delta\varepsilon^{\mathrm{t}}_{\mathrm{m}}\right) + \frac{\Delta\mathrm{E}}{\mathrm{E}}\Sigma_{\mathrm{m}} - \rho\alpha_{\mathrm{m}} - \frac{1}{\tilde{\alpha}_{\mathrm{m}}}\left(\frac{\Delta\tilde{\alpha}_{\mathrm{m}}}{\Delta\varepsilon_{\mathrm{m}}}\right)\rho\alpha_{\mathrm{m}}\Delta\theta, \tag{69}$$

and the equation Eq.(59).

$$\frac{3\mu' + \rho\left(\frac{\Delta\tilde{\alpha}_{\mathrm{eq}}}{\Delta\varepsilon_{\mathrm{eq}}}\right)_T}{3\lambda' + 2\mu' + \rho\left(\frac{\Delta\tilde{\alpha}_{\mathrm{m}}}{\Delta\varepsilon_{\mathrm{m}}}\right)_T}\cdot\frac{\Sigma^*_{\mathrm{m}} - (\Sigma'_{\mathrm{m}} - \rho\alpha'_{\mathrm{m}})}{\Sigma^*_{\mathrm{eq}} - \Sigma'_{\mathrm{eq}}} =$$
$$\frac{\tilde{p}''}{2}\frac{\left[\rho\Sigma'_0 + (1-\rho)\Sigma'_1\right]^2}{\left[\rho\Sigma'_0 + (1-\rho)\Sigma'_2\right]\Sigma'_{\mathrm{eq}}}\,\mathrm{sh}\left(\frac{3}{2}\frac{\Sigma'_{\mathrm{m}} - \rho\alpha'_{\mathrm{m}}}{\rho\Sigma'_0 + (1-\rho)\Sigma'_2}\right).$$

The system of the two equations Eq.(20)$_1$ and Eq.(30) is solved with respect to the unknowns $\sigma'_{\mathrm{eq}}$ and $'_{\mathrm{m}} - \rho\alpha'_{\mathrm{m}}$ by the same method as in the isotropic case ( equations Eq.(31), Eq.(32 ), Eq.(33), Eq.(34) ), with the substitutions

- $\Sigma'_1 \to \rho\Sigma'_0 + (1-\rho)\Sigma'_1$
- $\Sigma'_2 \to \rho\Sigma'_0 + (1-\rho)\Sigma'_2$,
- $3\mu \to 3\mu' + \rho\left(\Delta\tilde{\alpha}_{\mathrm{eq}}/\Delta\varepsilon_{\mathrm{eq}}\right)_{\mathrm{T}}$,
- $3\lambda + 2\mu \to 3\lambda' + 2\mu' + \rho\left(\Delta\tilde{\alpha}_{\mathrm{m}}/\Delta\varepsilon_{\mathrm{m}}\right)_{\mathrm{T}}$,
- $\Sigma'_{\mathrm{m}} \to \Sigma'_{\mathrm{m}} - \rho\alpha'_{\mathrm{m}}$.

The rest of the resolution is the same as in the kinematic case. Note that in practice, the substitutions:

- $\Sigma'_1 \rightarrow \rho\Sigma'_0 + (1-\rho)\sigma'_1$ ,
- $\Sigma'_2 \rightarrow \rho\Sigma'_0 + (1-\rho)\sigma'_2$,
- $\tilde{\alpha}_{\mathrm{eq}} \rightarrow \rho\tilde{\alpha}_{\mathrm{eq}}$:
- $\tilde{\alpha}_{\mathrm{m}} \rightarrow \rho\tilde{\alpha}_{\mathrm{m}}$.

*3.4. Particular Cases*

The first particular case is that, classic, of the elastic discharge: if the quantity

$$\frac{\Sigma^{*2}\mathrm{eq}}{\Sigma_1'^2} + 2\tilde{\mathrm{p}}\prime\ \mathrm{ch}\left(\frac{3}{2}\frac{\Sigma^{*}\mathrm{m}}{\Sigma'_2}\right) - 1 - \tilde{\mathrm{p}}'^2$$

(or the analogous quantities if work hardening is kinematic or mixed) is negative, there is discharge, therefore $\Delta\varepsilon_{\mathrm{eq}} = \Delta\varepsilon_{\mathrm{m}}^{\mathrm{p}} = \Delta\varepsilon_{\mathrm{m}} = 0, \Delta f_1 = 0$ and there is no need to perform constraint correction. The second is that of the closing of the cavities. Examining this possibility requires comparing the porosity to its initial value $f_0$. Given the explicit nature of the algorithm used with respect to this parameter, it makes sense to test not the true porosity $f'$ at time $\mathrm{t} + \Delta\mathrm{t}$ (which is known only at the end of the computation, after the convergence of the double iterative process), but on its approximation $\tilde{f}'$ given by Eq.(34) . The reclosing test is therefore the conjunction of the inequalities $\tilde{f}' \leq f_0$ and sh $< 0$, where sh denotes the hyperbolic sine of $\frac{3}{2}\frac{\sigma^{*}\mathrm{m}}{\sigma'_2}$ or analogous quantities. If this test is carried out, it is considered that the criterion is that of Von Mises and the flow rule, that naturally associated ($\Leftrightarrow \tilde{\mathrm{p}}' = \tilde{\mathrm{p}}'' = 0$ in the previous equations).

The third special case, in a way diametrically opposed to the previous one, is that of total damage, that is to say the one where the porosity becomes so high that p exceeds 1. In this case, the material is totally ruined. It is then enough, instead of performing the constraint correction as indicated above, to cancel $\mathbf{\Sigma}'$. The calculation of the evolution of the hardening parameters is not necessary, the material remaining by hypothesis ruined later[6], but it is necessary all the same to continue to calculate the local increment of porosity $\Delta f_1$, because it influences, in the event of relocation of the damage, the evolution of the porosity at the close points, the knowledge of which remains *a priori* necessary because these points may not themselves be ruined.

The last special case is that of sudden germination (decohesion of the metallic matrix around the inclusions). To treat this case, it is necessary to maintain at 0 the porosity (even, in the case of the delocalization of the damage, if that of the neighboring points already evolves) as long as the criterion Eq.(13) is not carried out. As soon as it becomes so, it is necessary to set $f = f_0$ and to continue the calculation normally.

[6]For this purpose, in the program, $f$ is prevented from decreasing again if $\mathrm{p}$ has reached or exceeded the value 1.

*3.5. Numerical treatment of the damage delocalization*

This proceudre uses an array $\mathrm{AF(I, J)}$. The first index varies from 1 to 6, the second from 1 to the total number of Gauss points concerned by the delocalization (it identifies the Gauss point). The meanings of the different quantities $\mathrm{AF(I, J)}$ are as follows:

- $\mathrm{AF(1,\ J), AF(2, J), AF(3, J)}$: Current coordinates of Gaussian point $\mathrm{J}$;
- $\mathrm{AF(4,\ J)}$ : Local porosity increment (between times $\mathrm{t}$ and $\mathrm{t} + \Delta\mathrm{t}$ ) at the Gaussian point $\mathrm{J}$;
- $\mathrm{AF(5, J)}$: Real increment (after convolution) of porosity at the Gaussian point $\mathrm{J}$;
- $\mathrm{AF(6, J)}$: Gauss point weight (for integration).

The calculation procedure is as follows: at all the iterations and for all the Gauss points, the a program is used to calculate the coordinates and the weight of the Gauss point and stores them in $AF(1-3, \mathrm{J})$ and $\mathrm{AF(6, J)}$. It also calls the a sub-program, which evaluates

the local porosity increment; the latter is stored in $\mathrm{AF(4,\ J)}$. Once the convergence on the nodal imbalances has been obtained, another program is called which, thanks to a double loop on the Gauss points, performs the convolution operation. The actual porosity increment at the point $J$, stored in $\mathrm{AF(5,\ J)}$, is transmitted to a program, which performs the final operation of calculating and storing the porosity at time $t$ and $t + \Delta t$ .

*3.6. Correction of the mean part of the deformation rate*

The first tests of the program made appear a difficulty which is not specific to the ductile fracture but arises in a general way in elastoplasticity large deformations. This difficulty consists of an inaccuracy in the calculation of the average part of the rate of total deformation (which affects, via the law of elasticity or the law of plastic flow in the case of the ductile damage, the average stress). The origin of this inaccuracy is as follows. Between two times of calculation $\mathrm{t}$ and $\mathrm{t} + \Delta\mathrm{t}$, the algorithm employed uses a formulation linearized compared to the increment of displacement $\Delta\mathbf{u}$; thus the increment of deformation is given by the formula:

$$\Delta\varepsilon_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial\Delta\mathrm{u_i}}{\partial\mathrm{x_j}} + \frac{\partial\Delta\mathrm{u_j}}{\partial\mathrm{x_i}}\right) \tag{70}$$

where the $\mathrm{x_i}$ designate the coordinates at time $\mathrm{t} + \Delta\mathrm{t}$. Similarly, the average strain is taken equal to

$$\Delta E_{\mathrm{m}} = \frac{1}{3}\frac{\partial\Delta\mathrm{u_i}}{\partial\mathrm{x_i}}. \tag{71}$$

The problem stems from the fact that due to quasi-incompressibility (compressibility is only due to elasticity and possibly damage, which, at least at the beginning of mechanical history, is weak), $\Delta E_{\mathrm{m}}$ is small compared to each of $\Delta\varepsilon_{\mathrm{ij}}$. As a result, the neglected second-order terms in the above formulas, although indeed small compared to each of $\partial\Delta\mathrm{u_i}/\partial\mathrm{x_j}$, are not small compared to the sum $\partial\Delta\mathrm{u_i}/\partial\mathrm{x_i}$, and that it is therefore illegal to delete them in the expression of $\Delta E_{\mathbf{m}}$.

We have therefore decided to calculate the deviatoric part $\Delta\mathbf{e}$ of the total deformation increment using a linearized formula, but its average part $\Delta E_m$ exactly . To do this, we evaluate the variation in volume between the instants t and $t + \Delta t$ using the exact formula:

$$\frac{v}{v + \Delta v} = \det\left(\delta_{ij} - \frac{\partial \Delta u_i}{\partial x_j}\right) \tag{72}$$

[7] and then $\Delta E_m$ by

$$3\Delta E_m = \frac{\Delta v}{v} = \frac{1}{\det\left(\delta_{ij} - \frac{\partial \Delta u_i}{\partial x_j}\right)} - 1 \tag{74}$$

This formula, linearized with respect to $\Delta v/v$, poses no problem because $\Delta v/\dot{v}$ is effectively small (only the expansion of $\Delta v/v$ to the first order according to the $\partial \Delta u_i/\partial x_j$, that the we are careful here not to perform, would pose one).

[7] It would seem more natural to use the formula instead:

$$\frac{v + \Delta v}{v} = \det\left(\delta_{ij} + \frac{\partial \Delta u_i}{\partial X_j}\right) \tag{73}$$

where the $x_j$ denote the coordinates at time t. But this would be more delicate because in practice, when passing from instant t to instant $t + \Delta t$ , only the coordinates $\left(x_j\right)$ at time $t + \Delta t$ (and associated shape functions), and not of those $\left(X_j\right)$ at time *t*.

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