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A projection algorithm for a constitutive model of ductile failure in porous plastic metals incorporating void shape effects

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1. INTRODUCTION

The famous Gurson's model [2] for the behaviors of ductile porous metals has already demonstrated its promise to predict several fracture problems involving ductile cracking. When incorporated in a finite element code it has preadicted the cup cone fracture phenomena of a round axisymmetric specimen ([4] and [3].)

A drawback of this model is that it assumes that the existing cavities in the materials are spherical, neglecting hereby the effects of the cavities shape on the general mechanical behavior of the material. This is a good approaximation when the triaxilaity (ratio of the mean stress with respect to the equivalent von Mises stress) is very high; indeed, the mean stress is then more higher than the mean deviatoric stress such that the cavity has the tendency to grow in the same way in all of the directions. However, when the triaxiality is small the effects of the cavities' shape become important and Gurson model predcits worse outcomes. For example, in a uniaxial tensile test, the triaxiality is $1/3$, and Gurson's model predicts a continuous increase of the porosity leading to coalescence of cavities and eventual failure.

In reality, however, the cavities quickly become cylindrical under the effect of the tensile stress exerted and the porosity then ceases to grow, its growth being governed by the lateral constraint which is zero, and not the average stress (which is positive) as the Gurson model predicts. Another example where the shape effects of the cavities are important is the case where the cavities are generated in the form of cracks (very flattened voids), for example by the rupture of a brittle phase in the material.

Gologanu *et al.* [1] have extended Gurson's model to include cavity shape effects. This led them to introduce a new parameter into the model, the shape factor of the cavities (related to the ratio of the axes of the voids, assumed to be axisymmetric ellipsoidal). The objective of this short note is to present the numerical implementation of this model into a finite element code. We first present a summary of the theoretical equations of the model and next, we describe the equations of the projection algorithm onto the yield surface the model involves.

2. THEORETICAL EQUATIONS OF THE MODEL

The voids are assumed ellipsoidal axisymmetric and aligned, of axe $\mathbf{e}_3$. The porosity f is defined as the ratio of the voids to the total volume of matter and voids. The shape factor of the cavity is defined as the logarithm of the ratio of the axes of a cavity according to $\mathbf{e}_3$ and a perpendicular direction.

2.1. PLASTICTY CRITERION, EVOLUTION EQUATIONS OF THE INTERNAL PARAMETERS

The yield criterion,which depends on the porosity f, the parameter of the shape factor S and a strain hardening parameter $\overline{\sigma}$ representing some "average of the elastic limit of the sound matrix," is writen as

$$\begin{aligned}\Phi(\underline{\underline{\sigma}}, f, S, \overline{\sigma}) \equiv &\frac{C}{\overline{\sigma}^2}\|\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}\|^2 + 2q(g+1)(g+f)cosh(K\frac{\sigma_h}{\overline{\sigma}}) \\ &- (g+1)^2 - q^2(g+f)^2 = 0.\end{aligned} \quad (1)$$

In this expression, $\boldsymbol{\sigma}'$represents the deviatoric stress $\boldsymbol{\sigma}$, $\mathbf{X}$ the tensor defined as

$$\boldsymbol{X} = \frac{1}{3}(-\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + 2\boldsymbol{e}_3 \otimes \boldsymbol{e}_3) \quad (2)$$

(where we recall that the vecteur $\mathbf{e}_3$ is parallel to the axes of the voids), and |||| the von Mises norm:

$$\|\boldsymbol{T}\| \equiv \left(\frac{3}{2}\boldsymbol{T} : \boldsymbol{T}\right)^{1/2} = \left(\frac{3}{2}\boldsymbol{T}_{ij}\boldsymbol{T}_{ij}\right)^{1/2}; \quad (3)$$

we shall adopt in the rest of this note the simplified notation

$$\sigma_{eq} \equiv \|\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}\| \quad (4)$$

(note that with this definition , $\sigma_{eq} \neq ||\boldsymbol{\sigma}'||$ as it is customary).

The paprameters C, η, K, g depend uniquely on f and S; their expressions are given in Gologanu *et al.* [1] but are not necessary here. In addition, the stress σ_h worths

$$\sigma_h = \alpha_2(\sigma_{11} + \sigma_{22}) + (1 - 2\alpha_2)\sigma_{33} \quad (5)$$

where here also, $\alpha 2$ is a parameter which depends uniquely on f and S whose expressions are given in Gologanu *et al.* [1]. Finally, q is the "Tvergaard's parameter [**?**]" whose value depends here of the shape of the cavities given in Gologanu *et al.* [1].

The law of evolution of porosity is classically deduced from the approximited incompressibility (the elasticity being neglected) of the sound matrix:

$$\dot{f} = 3(1-f)\dot{\varepsilon}^p_m \quad (6)$$

where $\dot{\varepsilon}_m^p = \frac{1}{3}tr(\dot{\boldsymbol{\varepsilon}}^p)$ represents the mean part of the strain rate $\dot{\boldsymbol{\varepsilon}}^p$. The law of the evolution of the shape factor of cavities is as follows

$$\dot{S} = \frac{3}{2}h\dot{\varepsilon}_{33}^{p'} + 3\left(\frac{1-3\alpha_1}{f} + 3\alpha_2 - 1\right)\dot{\varepsilon}_m^p \tag{7}$$

where $\dot{\boldsymbol{\varepsilon}}^{p'}$ represents the deviatoric strain rate of the plastic deformation $\dot{\boldsymbol{\varepsilon}}^p$, $\dot{\varepsilon}^p = \frac{1}{3}tr\dot{\boldsymbol{\varepsilon}}^p$ its mean part as above and α_1 a parameter depending here also on the parameters f and S and whose precise expression is given in Gologanu *et al.* [1]. Finally, h is a parameter depending, in addition to f and S, on the triaxility T defined by

$$T = \frac{\sigma_m}{\|\boldsymbol{\sigma}'\|} \tag{8}$$

where $\sigma_m = \frac{1}{3}tr\boldsymbol{\sigma}$ is the mean part of the stress tensor. The parameter $\overline{\sigma}$ is given as a function of a strain hardening parameter $\overline{\varepsilon}$ representing roughly the average equivalent deformation of the sound matrix by the formula

$$\overline{\sigma} = \sigma(\overline{\varepsilon}) \tag{9}$$

where $\sigma(\overline{\varepsilon}$ represents the function given the Cauchy stress as a function of the logarithmic deformation in a simple tensile test on the sound material. The evolution equation of $\overline{\varepsilon}$ is the same as the one proposed by Gurson

$$(1-f)\overline{\sigma}\;\dot{\overline{\varepsilon}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p. \tag{10}$$

Finally, the evolution equation of $\mathbf{x}_3$ parallel to the axis of the voids is given by

$$\dot{\boldsymbol{e}}_3 = \boldsymbol{\Omega} \cdot \boldsymbol{e}_3 \tag{11}$$

where $\boldsymbol{\Omega}$ is the "rotation rate of the matter" (for example the antisymmetric part of the velocity gradient.) This equation is based on the heuristic hypothesis that the voids and the matter has the same rotation rate.

2.2. FLOW RULE ASSOCIATED TO THE YILED CRITERION BY NORMALITY

As usual, we assume the partition of the total deformation rate $\dot{\boldsymbol{\varepsilon}}$ between the elastic deformation rate $\dot{\boldsymbol{\varepsilon}}^e$ and the plastic formation rate $\dot{\boldsymbol{\varepsilon}}^p$. The first rate is given by the usual elasticity law and the second by

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial\Phi}{\partial\boldsymbol{\sigma}}, \quad \dot{\lambda} \geq 0 \tag{12}$$

where Φ represents (remember) the yield function and the plastic multiplier $\dot{\lambda}$. It is now a question of explaining this equation. To do this, let's start by evaluating

$$
\begin{aligned}
\frac{\partial \sigma_{eq}^2}{\partial \sigma_{ij}} =& \frac{\partial}{\partial \sigma_{ij}} \left[\frac{3}{2} \left(\sigma'_{kl} + \eta \sigma_h X_{kl} \right) \left(\sigma'_{kl} + \eta \sigma_h X_{kl} \right) \right] \\
&= 3 \left(\sigma'_{kl} + \eta \sigma_h X_{kl} \right) \left(\delta_{ik} \delta_{jl} - \frac{1}{3} \delta_{ij} \delta_{kl} + \eta \frac{\partial \sigma_h}{\partial \sigma_{ij}} X_{kl} \right) \\
&= 3 \left[\sigma'_{ij} + \eta \sigma_h X_{ij} + \left(\sigma'_{kl} + \eta \sigma_h X_{kl} \right) \eta \frac{\partial \sigma_h}{\partial \sigma_{ij}} X_{kl} \right] \\
&= 3 \left[\sigma'_{ij} + \eta \sigma_h X_{ij} + \frac{2}{3} \eta \frac{\partial \sigma_h}{\partial \sigma_{ij}} \left(\frac{3}{2} \boldsymbol{\sigma'} : \boldsymbol{X} + \eta \sigma_h \right) \right].
\end{aligned}
$$

As a result, the Greek indices taking only values 1 and 2 and taking into account the obvious relationships $\partial \sigma_h \Big/ \partial \sigma_{\alpha\beta} = \alpha_2 \delta_{\alpha\beta}$, $\partial \sigma_h \Big/ \partial \sigma_{\alpha 3} = 0$, $\partial \sigma_h \Big/ \partial \sigma_{33} = 1 - 2\alpha_2$, we get

$$
\begin{aligned}
\frac{\partial \Phi}{\partial \sigma_{\alpha\beta}} =& \frac{3C}{\overline{\sigma}^2} \left[\sigma'_{\alpha\beta} + \eta \sigma_h X_{\alpha\beta} + \frac{2}{3} \eta \alpha_2 \delta_{\alpha\beta} \left(\frac{3}{2} \boldsymbol{\sigma'} : \boldsymbol{X} + \eta \sigma_h \right) \right] \\
& + 2q(g+1)(g+f) \frac{K}{\overline{\sigma}} \alpha_2 \delta_{\alpha\beta} sinh \left(K \frac{\sigma_h}{\overline{\sigma}} \right); \\
\frac{\partial \Phi}{\partial \sigma_{\alpha 3}} =& \frac{3C}{\overline{\sigma}^2} \sigma_{\alpha 3}; \qquad (13) \\
\frac{\partial \Phi}{\partial \sigma_{33}} =& \frac{3C}{\overline{\sigma}^2} \left[\sigma'_{33} + \eta \sigma_h X_{33} + \frac{2}{3} \eta (1 - 2\alpha_2) \left(\frac{3}{2} \boldsymbol{\sigma'} : \boldsymbol{X} + \eta \sigma_h \right) \right] \\
& + 2q(g+1)(g+f) \frac{K}{\overline{\sigma}} (1 - 2\alpha_2) sinh \left(K \frac{\sigma_h}{\overline{\sigma}} \right).
\end{aligned}
$$

As a consequence

$$
\begin{aligned}
\dot{\varepsilon}_m^p =& \frac{1}{3} \left(\dot{\varepsilon}_{\alpha\alpha}^p + \dot{\varepsilon}_{33}^p \right) = \frac{\dot{\lambda}}{3} \left(\frac{\partial \Phi}{\partial \sigma_{\alpha\alpha}} + \frac{\partial \Phi}{\partial \sigma_{33}} \right) \\
=& \frac{\dot{\lambda}}{3} \left[\frac{2C\eta}{\overline{\sigma}^2} \left(\frac{3}{2} \boldsymbol{\sigma'} : \boldsymbol{X} + \eta \sigma_h \right) + 2q(g+1)(g+f) \frac{K}{\overline{\sigma}} sinh \left(K \frac{\sigma_h}{\overline{\sigma}} \right) \right]. \qquad (14)
\end{aligned}
$$

In addition, combining Eqs. (12, 13, 14), we see that

$$\dot{\varepsilon}^p_{\alpha\beta} =\dot{\lambda}\cdot\frac{3C}{\overline{\sigma}^2}\left(\sigma'_{\alpha\beta}+\eta\sigma_h X_{\alpha\beta}\right)+\alpha_2\delta_{\alpha\beta}\cdot 3\dot{\varepsilon}^p_m;$$

$$\dot{\varepsilon}^p_{\alpha 3} =\dot{\lambda}\cdot\frac{3C}{\overline{\sigma}^2}\sigma_{\alpha 3}; \tag{15}$$

$$\dot{\varepsilon}^p_{33} =\dot{\lambda}\cdot\frac{3C}{\overline{\sigma}^2}\left(\sigma'_{33}+\eta\sigma_h X_{33}\right)+(1-2\alpha_2)\cdot 3\dot{\varepsilon}^p_m.$$

Let assume that

$$\dot{\boldsymbol{\varepsilon}}^p_d =\dot{\boldsymbol{\varepsilon}}^p-3\alpha_2\dot{\varepsilon}^p_m\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\alpha-3(1-2\alpha_2)\dot{\varepsilon}^p_m\boldsymbol{e}_3\otimes\boldsymbol{e}_3 \tag{16}$$

(It will be observed that $\dot{\boldsymbol{\varepsilon}}^p_d$ is a pure deviator, that is to say that $tr\dot{\boldsymbol{\varepsilon}}^p_d=0$). From Eqs. (15, 16)

$$\dot{\boldsymbol{\varepsilon}}^p_d =\dot{\lambda}\cdot\frac{3C}{\overline{\sigma}^2}\left(\boldsymbol{\sigma'}+\eta\sigma_h\boldsymbol{X}\right).$$

Thus, the tensors $\dot{\boldsymbol{\varepsilon}}^p_d$ and $\boldsymbol{\sigma'}+\eta\sigma_h\boldsymbol{X}$ are positively colinear. We immediately deduce that

$$\dot{\boldsymbol{\varepsilon}}^p_d =\frac{3}{2}\frac{\dot{\varepsilon}^p_d}{\sigma_{eq}}\left(\boldsymbol{\sigma'}+\eta\sigma_h\boldsymbol{X}\right). \tag{17}$$

where

$$\dot{\varepsilon}^p_d=\left(\frac{2}{3}\dot{\boldsymbol{\varepsilon}}^p_d:\dot{\boldsymbol{\varepsilon}}^p_d\right)^{1/2} \tag{18}$$

($\dot{\varepsilon}^p_d$ worths the von Mises's norm of $\dot{\boldsymbol{\varepsilon}}^p_d$) and σ_{eq} is given by Eq.(4). In addition, we immediatly obtain

$$\dot{\lambda}=\frac{1}{2C}\frac{\overline{\sigma}^2\dot{\varepsilon}^p_d}{\sigma_{eq}};$$

reporting this result in Eq.(14), we get

$$\frac{\dot{\varepsilon}^p_m}{\dot{\varepsilon}^p_d}=\frac{1}{6C}\frac{\overline{\sigma}^2}{\sigma_{eq}}\left[\frac{2C\eta}{\overline{\sigma}^2}\left(\frac{3}{2}\boldsymbol{\sigma'}:\boldsymbol{X}+\eta\sigma_h\right)+2q(g+1)(g+f)\frac{K}{\overline{\sigma}}sinh\left(K\frac{\sigma_h}{\overline{\sigma}}\right)\right]$$

or

$$\frac{\dot{\varepsilon}_m^p}{\dot{\varepsilon}_d^p} = \frac{\eta}{3\sigma_{eq}}\left(\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h\right) + q(g+1)(g+f)\frac{K}{3C}\frac{\overline{\sigma}}{\sigma_{eq}} sinh\left(K\frac{\sigma_h}{\overline{\sigma}}\right). \tag{19}$$

The equations Eqs. (17, 19) (where $\dot{\boldsymbol{\varepsilon}}_d^p$ is defined by Eq.(16) and $\dot{\varepsilon}_d^p$ by Eq.(18)) consists of the plastic flow rule of the material.

3. NUMERICAL IMPLEMENTATION

3.1. PROJECTION ON THE YIELD SURFACE

The essential problem of any numerical implementation of an elastic plastic model is that of the projection onto the yield surface. This problem is as follows: from the results of a " large elastic-plastic iteration" (elastic resolution over the whole structure with initial plastic deformations given), which provides the increment of total deformations $\nabla\dot{\varepsilon}$ between the time t and $t + \nabla t$ of the calculation, find the decompostion of $\dot{\varepsilon}$ into elastic $\dot{\varepsilon}^e$ and plastic $\dot{\varepsilon}^p$ (using the yield criterion at $t + \nabla t$ and the flow rule (between t and $t + \nabla t$)) and the stress at $t + \nabla t$.

In the subsequent, the quantities without indices are taken at the moment $t + nablat$ while those with an index "0" will be taken at the time t (it is therefore a question of known quantities.)

Let us begin, as in the case of the original Gurson criterion, by defining a parameterization of the original Gurson criterion of the criterion by means of an angle phi, ensuring automatic satisfaction. The flow rules we will then provide an equation on ϕ which can be resolved numerically.

To find this parameterization, let us look for the maximum value of $C\frac{\sigma_{eq}^2}{\overline{\sigma}^2}$ corresponding to $\sigma_h = 0 \implies \cosh\left(K\frac{\sigma_h}{\overline{\sigma}}\right) = 1$; according to Eq.(1)

$$C\frac{\sigma_{eq}^2}{\overline{\sigma}^2} = (g+1)^2 + q^2(g+f)^2 - 2q(g+1)(g+f) = [g+1-q(g+f)]^2.$$

It is therefore natural to assume that

$$\begin{aligned} C\frac{\sigma_{eq}^2}{\overline{\sigma}^2} &= [g+1-q(g+f)]^2 cos^2\varphi \\ \Rightarrow \sigma_{eq} &= \frac{\overline{\sigma}}{\sqrt{C}}[g+1-q(g+f)]cos\varphi \end{aligned} \tag{20}$$

where ϕ is some angle with positive cosine. We get from Eq.(1)

$$
\begin{aligned}
2q(g+1)(g+f)cosh\left(K\frac{\sigma_h}{\overline{\sigma}}\right) &= (g+1)^2+q^2(g+f)^2-[g+1-q(g+f)]^2cos^2\varphi \\
&= (g+1)^2+q^2(g+f)^2-[g+1-q(g+f)]^2 \\
&\quad +[g+1-q(g+f)]^2sin^2\varphi \\
&= 2q(g+1)(g+f)+[g+1-q(g+f)]^2sin^2\varphi
\end{aligned}
$$

$$
\Rightarrow cosh\left(K\frac{\sigma_h}{\overline{\sigma}}\right) = 1+\frac{[g+1-q(g+f)]^2}{2q(g+1)(g+f)}sin^2\varphi
$$

$$
\Rightarrow \sigma_h = \frac{\overline{\sigma}}{K}sgn(\varphi)cosh^{-1}\left(1+\frac{[g+1-q(g+f)]^2}{2q(g+1)(g+f)}sin^2\varphi\right) \tag{21}
$$

where we introduce a "sgn(ϕ)" (sign of ϕ) to allow σ_h to take all possible values, both negative and positive. The Eqs. (20, 21) is the parameterization of the criterion we are looking for. The interval of variation of the angle ϕ can be taken equal to $\left[-\frac{\pi}{2},+\frac{\pi}{2}\right]$: it allows $cos(\phi)$ to take all positive or zero values, as well as $sgn(\phi)$ to take values ± 1. Before writing the flow rule in a discretized form, let us begin by establishing the relationships that exist between σ_h and σ_m on one hand, $\dot{\boldsymbol{\varepsilon}}_d^p$ and $\dot{\boldsymbol{\varepsilon}}^{p'}$ on the other hand. First, we get Eqs. (5)

$$
\begin{aligned}
\sigma_h =& \alpha_2\sigma_{\alpha\alpha}+(1-2\alpha_2)\sigma_{33} = \frac{1}{3}(\sigma_{\alpha\alpha}+\sigma_{33})+\left(\alpha_2-\frac{1}{3}\right)\sigma_{\alpha\alpha}+2\left(\frac{1}{3}-\alpha_2\right)\sigma_{33} \\
=& \sigma_m+(1-3\alpha_2)\left(-\frac{1}{3}\sigma_{\alpha\alpha}+\frac{2}{3}\sigma_{33}\right)
\end{aligned}
$$

which gives, from the definition Eq. (2) of the tensor $\boldsymbol{X}$:

$$
\sigma_h = \sigma_m+(1-3\alpha_2)\boldsymbol{\sigma'}:\boldsymbol{X}. \tag{22}
$$

In addition, from Eq. (16),

$$
\begin{aligned}
\dot{\boldsymbol{\varepsilon}}_d^p =& \dot{\boldsymbol{\varepsilon}}^p-3\alpha_2\dot{\varepsilon}_m^p\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\alpha-3(1-2\alpha_2)\dot{\varepsilon}_m^p\boldsymbol{e}_3\otimes\boldsymbol{e}_3 \\
=& \dot{\boldsymbol{\varepsilon}}^p-\dot{\varepsilon}_m^p\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\alpha-\dot{\varepsilon}_m^p\boldsymbol{e}_3\otimes\boldsymbol{e}_3+(1-2\alpha_2)\dot{\varepsilon}_m^p\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\alpha+2(3\alpha_2-1)\dot{\varepsilon}_m^p\boldsymbol{e}_3\otimes\boldsymbol{e}_3 \\
=& \dot{\boldsymbol{\varepsilon}}^p-3(1-3\alpha_2)\dot{\varepsilon}_m^p\boldsymbol{X}.
\end{aligned} \tag{23}
$$

Let us now write the flow rule in discrtized form. $\Delta\varepsilon_d^p$ being related to $\Delta\varepsilon^{p'}$ par the relationship

$$\Delta\boldsymbol{\varepsilon}_d^p = \Delta\boldsymbol{\varepsilon}^{p'} - 3(1-3\alpha_2)\Delta\varepsilon_m^p \boldsymbol{X} \tag{24}$$

(which is the discritized equivalent form of Eq.(23), we get (see Eqs.(17) and Eq.(18))

$$\Delta\boldsymbol{\varepsilon}_d^p = \frac{3}{2}\frac{\Delta\boldsymbol{\varepsilon}_d^p}{\sigma_{eq}}\left(\boldsymbol{\sigma'} + \eta\sigma_h \boldsymbol{X}\right) \tag{25}$$

where

$$\Delta\boldsymbol{\varepsilon}_d^p = \left(\frac{3}{2}\Delta\boldsymbol{\varepsilon}_d^p : \Delta\boldsymbol{\varepsilon}_d^p\right)^{1/2}. \tag{26}$$

Note that these equations correspond to an implicit algorithm with respect to all parameters except the porosity f. The symbol $\tilde{f}$ represents an explicit approximation of porosite on the half-interval $t + \frac{\Delta t}{2}$ given by

$$\frac{\Delta\varepsilon_m^p}{\Delta\varepsilon_d^p} = \frac{\eta}{3\sigma_{eq}}\left(\frac{3}{2}\boldsymbol{\sigma'} : \boldsymbol{X} + \eta\sigma_h\right) + q(g+1)(g+\tilde{f})\frac{K}{3C}\frac{\overline{\sigma}}{\sigma_{eq}} sinh\left(K\frac{\sigma_h}{\overline{\sigma}}\right). \tag{27}$$

The explicit character of the algorithm with respect to f (parameter governing softening) ensures its convergence, taking $\tilde{f}$ at $t + \frac{\Delta t}{t}$ and not at t or $t + \Delta t$ allow to optimize the precision of the algorithm.

$$\tilde{f} = f_0 + \dot{f}_0\frac{\Delta t}{2}. \tag{28}$$

Assume $\boldsymbol{\sigma}'_o$ and $\sigma_m^\star$ the deviatoric and the mean parts of the stresses tensor (at $t + \Delta t$) $\boldsymbol{\sigma}^\star$ "elastically calculated," that is by assuming that the increment of deformation $\Delta\boldsymbol{\varepsilon}$ is purely elastic; we get

$$\boldsymbol{\sigma}^{*'} = \boldsymbol{\sigma}'_o + 2\mu\Delta\boldsymbol{\varepsilon}' \quad \text{and} \quad \sigma_m = \sigma_{mo} + (3\lambda + 2\mu)\,\Delta\varepsilon_m. \tag{29}$$

$\boldsymbol{\sigma}'_o$ and $\sigma_m^\star$ are the known quantities during the operation of the "projection on the yield surface." Now let evaluate $\boldsymbol{\sigma'}$ using Eqs.(**??**) , Eqs.(24), Eq.(25))

$$
\begin{aligned}
\boldsymbol{\sigma}' =& \boldsymbol{\sigma}'_o + 2\mu\Delta\boldsymbol{\varepsilon}^{e'} = \boldsymbol{\sigma}'_o + 2\mu\Delta\boldsymbol{\varepsilon}' - 2\mu\Delta\boldsymbol{\varepsilon}^{p'} = \boldsymbol{\sigma}^{*'} - 2\mu\Delta\boldsymbol{\varepsilon}^{p'} \\
=& \boldsymbol{\sigma}^{*'} - 2\mu\Delta\boldsymbol{\varepsilon}^p_d - 6\mu(1-3\alpha_2)\Delta\varepsilon^p_m\boldsymbol{X} \\
=& \boldsymbol{\sigma}^{*'} - 3\mu\frac{\Delta\varepsilon^p_d}{\sigma_{eq}}\left(\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}\right) - 6\mu(1-3\alpha_2)\Delta\varepsilon^p_m\boldsymbol{X}.
\end{aligned}
\tag{30}
$$

Contracting this equation with the tensor $\frac{3}{2}\boldsymbol{X}$ using Eq.(27)

$$
k = q(g+1)(g+\tilde{f})\frac{K}{3C}\overline{\sigma}sinh\left(K\frac{\sigma_h}{\overline{\sigma}}\right); \tag{31}
$$

we get

$$
\begin{aligned}
\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} =& \frac{3}{2}\boldsymbol{\sigma}^{*'} : \boldsymbol{X} - 3\mu\frac{\Delta\varepsilon^p_d}{\sigma_{eq}}\left(\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h\right) \\
& -6\mu(1-3\alpha_2)\left[\frac{\eta}{3\sigma_{eq}}\left(\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h\right) + \frac{k}{\sigma_{eq}}\right]\Delta\varepsilon^p_d
\end{aligned}
$$

thus, adding $\eta\sigma_h$ to the two sides of the equations, we get:

$$
\begin{aligned}
\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h =& \frac{3}{2}\boldsymbol{\sigma}^{*'} : \boldsymbol{X} + \eta\sigma_h - 3\mu\frac{\Delta\varepsilon^p_d}{\sigma_{eq}}\left(\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h\right) \\
& - 2\mu\frac{\eta}{\sigma_{eq}}(1-3\alpha_2)\left(\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h\right)\Delta\varepsilon^p_d - 6\mu(1-3\alpha_2)\frac{k}{\sigma_{eq}}\Delta\varepsilon^p_d
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow & \left[1 + 3\mu\frac{\Delta\varepsilon^p_d}{\sigma_{eq}} + 2\mu\eta(1-3\alpha_2)\frac{\Delta\varepsilon^p_d}{\sigma_{eq}}\right]\left(\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h\right) \\
& = \frac{3}{2}\boldsymbol{\sigma}^{*'} : \boldsymbol{X} + \eta\sigma_h - 6\mu k(1-3\alpha_2)\frac{\Delta\varepsilon^p_d}{\sigma_{eq}}
\end{aligned}
$$

$$
\Rightarrow \frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h = \frac{a\Delta\varepsilon^p_d + b}{c\Delta\varepsilon^p_d + d} \tag{32}
$$

where

$$
\begin{aligned}
& a = -6\mu k(1-3\alpha_2) \quad ; \quad b = \left(\frac{3}{2}\boldsymbol{\sigma}^{*'} : \boldsymbol{X} + \eta\sigma_h\right)\sigma_{eq} \; ; \\
& c = 3\mu + 2\mu\eta(1-3\alpha_2) \quad ; \quad d = \sigma_{eq}.
\end{aligned}
\tag{33}
$$

Let's go back now to Eq.(30) by adding $\eta\sigma_h\boldsymbol{X}$ to the two sides of the equations;

we obtain

$$\begin{aligned}&\boldsymbol{\sigma}' + \eta\sigma_h \boldsymbol{X} = \boldsymbol{\sigma}^{*'} + \eta\sigma_h \boldsymbol{X} - 3\mu\frac{\Delta\varepsilon_d^p}{\sigma_{eq}}\left(\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}\right) - 6\mu(1-3\alpha_2)\Delta\varepsilon_m^p\boldsymbol{X}\\ &\Rightarrow \left(1 + 3\mu\frac{\Delta\varepsilon_d^p}{\sigma_{eq}}\right)\left(\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}\right) = \boldsymbol{\sigma}^{*'} + \eta\sigma_h\boldsymbol{X} - 6\mu(1-3\alpha_2)\Delta\varepsilon_m^p\boldsymbol{X}.\end{aligned}$$

In addition, by Eq.(30) we have

$$\begin{aligned}&\sigma_m = \sigma_{mo} + (3\lambda + 2\mu)\,\Delta\varepsilon_m^e = \sigma_m^* - (3\lambda+2\mu)\,\Delta\varepsilon_m^p\\ &\Rightarrow \Delta\varepsilon_m^p = \frac{\sigma_m^* - \sigma_m}{3\lambda + 2\mu},\end{aligned} \tag{34}$$

thus, by reporting in the previous equation, we get

$$\left(1 + 3\mu\frac{\Delta\varepsilon_d^p}{\sigma_{eq}}\right)\left(\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}\right) = \boldsymbol{\sigma}^{*'} + \eta\sigma_h\boldsymbol{X} - \frac{6\mu}{3\lambda+2\mu}(1-3\alpha_2)(\sigma_m^* - \sigma_m)\boldsymbol{X}. \tag{35}$$

Taking the von Mises norm $\|\|$ of the two sides of the equation, we get

$$\begin{aligned}&\sigma_{eq} + 3\mu\Delta\varepsilon_d^p = \left\|\boldsymbol{\sigma}^{*'} + \eta\sigma_h\boldsymbol{X} - \frac{6\mu}{3\lambda+2\mu}(1-3\alpha_2)(\sigma_m^*-\sigma_m)\boldsymbol{X}\right\|\\ &\Rightarrow \Delta\varepsilon_d^p = \frac{1}{3\mu}\left(\left\|\boldsymbol{\sigma}^{*'} + \eta\sigma_h\boldsymbol{X} - \frac{6\mu}{3\lambda+2\mu}(1-3\alpha_2)(\sigma_m^*-\sigma_m)\boldsymbol{X}\right\| - \sigma_{eq}\right).\end{aligned} \tag{36}$$

Finally, using the flow rule Eq.(27) together with Eqs.(31, 32, 34) we obtain

$$\begin{aligned}&\Delta\varepsilon_m^p = \frac{\sigma_m^*-\sigma_m}{3\lambda+2\mu} = \left(\frac{\eta}{3\sigma_{eq}}\frac{a\Delta\varepsilon_d^p + b}{c\Delta\varepsilon_d^p + d} + \frac{k}{\sigma_{eq}}\right)\Delta\varepsilon_d^p\\ &\Rightarrow \frac{\sigma_m^*-\sigma_m}{3\lambda+2\mu}\sigma_{eq} - \left(\frac{\eta}{3}\frac{a\Delta\varepsilon_d^p+b}{c\Delta\varepsilon_d^p+d} + k\right)\Delta\varepsilon_d^p = 0.\end{aligned} \tag{37}$$

Let us observe that $\Delta\varepsilon_d^p$ can be express as a function of ϕ and $\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X}$ thanks to Eq.(36), considering Eqs.(20, 21, 22). Thus, we can choose ϕ and $\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X}$ as principal unknowns. These equations satisfy Eqs.(32, 37) where the coefficients a, b, c, d are given by Eq.(**??**) (k itsef being given by Eq.(31)) .

These equations can be solved numerically by Newton's method: the quantity $\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X}$ can be evaluated by solving Eq.(32), ϕ being calculated at each Newton's iteration on $\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X}$ by solving Eq.(37) by Newton iteration on ϕ. Once ϕ and $\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X}$ are determined, we deduced σ_{eq}, σ_h, and σ_m by Eqs.(20, 21, 22), and $\Delta\varepsilon^p_m$ and $\Delta\varepsilon^p_d$ by Eqs.(34, 36), $\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}$ (and hence $\boldsymbol{\sigma}'$) by Eq.(35), $\Delta\boldsymbol{\varepsilon}^p_m$ by Eq.(25) and $\Delta\boldsymbol{\varepsilon}^{p\prime}$ by Eq.(24). Thus, the operation of projection onto the yield locus has been carried out.

3.2. EVOLUTION EQUATIONS FOR THE INTERNAL PARAMETERS

The first internal parameter is the porosity f. Adopting an implicit algorithm with respect to this quantity leads to unsolvable convergence problems. We therefore adopt an explicit algorithm where f (as it appears for example in Eqs.(20), 21) does not represent the true value of the porosity at time $t + \Delta t$ but the approximation given by

$$f = f_o + \dot{f}_o\Delta t \tag{38}$$

(f is therefore fixed throughout the passage from the instant t to instant $t + \Delta t$).

Of course, after convergence of the large elastic plastic iterations from t to $t + \Delta t$, f is updated for the next step thanks to the following formula, discretized equivalent of Eq.(6)

$$\Delta t = 3\left(1 - \tilde{f}\right)\Delta\varepsilon^p_m. \tag{39}$$

The (approximate) value $\tilde{f}$ of the porosity in the half-interval (see Eq.(**??**)) is used here in order to improve the accuracy of the algorithm.

The second internal parameter is the shape factor S, also unknown a priori. To determine it, we adopt an iterative algorithm of a "fixed point" type. The law of evolution of this parameter is the discretized equivalent of Eq.(7)

$$\Delta S = \frac{3}{2}h\Delta\varepsilon^{p\prime}_{33} + 3\left(\frac{1 - 3\alpha_1}{f} + 3\alpha_2 - 1\right)\Delta\varepsilon^p_m. \tag{40}$$

We recall that h is an independent parameter, besides f and S, of the triaxiality $\boldsymbol{T}$ defined by $Eq.(8)$. It is therefore necessary to calculate, in addition to σ_m as we saw above, $\|\boldsymbol{\sigma}'\|$, quantity which, we recall, is not equal to $\sigma_{eq} = \left(\|\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}\|\right)$. By definition of the von Mises norm $\|\|$ defined by

$$\begin{aligned}\sigma^2_{eq} &= \|\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}\|^2 = \frac{3}{2}\left(\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}\right) : \left(\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}\right)\\ &= \frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}' + 3\eta\sigma_h\boldsymbol{\sigma}' : \boldsymbol{X} + \eta^2 + \sigma^2_h = \|\boldsymbol{\sigma}'\|^2 + 2\eta\sigma_h\left(\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h\right) - \eta^2\sigma^2_h\end{aligned}$$

$$\Rightarrow \|\boldsymbol{\sigma}'\| = \left[\sigma_{eq}^2 - 2\eta\sigma_h\left(\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h\right) + \eta^2\sigma_h^2\right]^{1/2}. \tag{41}$$

This equation allows to evaluate $\|\boldsymbol{\sigma}'\|$ and therefore the triaxiality $\|\boldsymbol{T}\|$, the quantities σ_{eq}, σ_h, $\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h$ being known elsewhere.

The third internal parameter is the hardening parameter $\overline{\sigma}$, or what amounts to the same via Eq.(9), the mean equivalent deformation $\overline{\varepsilon}$. We use a fixed point algotihtm to calculate this paremeter, as for the shape form factor. The law of evolution used, discretized equivalent of Eq.(10), is

$$\left(1 - \tilde{f}\right)\overline{\sigma}\Delta\overline{\varepsilon} = \boldsymbol{\sigma} : \Delta\boldsymbol{\varepsilon}^p. \tag{42}$$

Its use requires the calculation of $\boldsymbol{\sigma} : \Delta\boldsymbol{\varepsilon}^p$ according to known quantities. We get, from Eqs.(22), 24) and Eq.(25),

$$\begin{aligned}
\boldsymbol{\sigma} : \Delta\boldsymbol{\varepsilon}^p &= (\boldsymbol{\sigma}' + \sigma_m\mathbf{1}) : \left(\Delta\boldsymbol{\varepsilon}^{p'} + \Delta\varepsilon_m^p\mathbf{1}\right) = \boldsymbol{\sigma}' : \Delta\boldsymbol{\varepsilon}^{p'} + 3\sigma_m\Delta\varepsilon_m^p \\
&= \boldsymbol{\sigma}' : (\Delta\boldsymbol{\varepsilon}_d^p + 3(1-3\alpha_2)\Delta\varepsilon_m^p\boldsymbol{X}) + 3\left(\sigma_h - (1-3\alpha_2)\boldsymbol{\sigma}' : \boldsymbol{X}\right)\Delta\varepsilon_m^p \\
&= \boldsymbol{\sigma}' : \Delta\boldsymbol{\varepsilon}_d^p + 3\sigma_h\Delta\varepsilon_m^p \\
&= \boldsymbol{\sigma}' : \frac{3}{2}\frac{\Delta\varepsilon_d^p}{\sigma_{eq}}(\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}) + 3\sigma_h\Delta\varepsilon_m^p \\
&= (\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}) : \frac{3}{2}\frac{\Delta\varepsilon_d^p}{\sigma_{eq}}(\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}) - \frac{3}{2}\frac{\Delta\varepsilon_d^p}{\sigma_{eq}}\eta\sigma_h\boldsymbol{X} : (\boldsymbol{\sigma}' + \eta\sigma_h\boldsymbol{X}) + 3\sigma_h\Delta\varepsilon_m^p \\
&= \sigma_{eq}\Delta\varepsilon_d^p + 3\sigma_h\Delta\varepsilon_m^p - \eta\frac{\sigma_h}{\sigma_{eq}}\left(\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h\right)\Delta\varepsilon_d^p
\end{aligned}$$

thus, the evolution equation of Eq.(42) of $\overline{\varepsilon}$ can be written as

$$\Delta\overline{\varepsilon} = \frac{1}{\left(1 - \tilde{f}\right)\overline{\sigma}}\left[\sigma_{eq}\Delta\varepsilon_d^p + 3\sigma_h\Delta\varepsilon_m^p - \eta\frac{\sigma_h}{\sigma_{eq}}\left(\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{X} + \eta\sigma_h\right)\Delta\varepsilon_d^p\right] \tag{43}$$

where all the quantities in the right side of the equation are known quantities.

The fourth internal parameter is the vector $\boldsymbol{e}_3$ parallel to the void axis. Its law of evolution Eq.(11) is discretized in an explicit way following the expression:

$$\Delta\boldsymbol{e}_3 = \Delta\boldsymbol{\Omega} \cdot (\boldsymbol{e}_3)_o \tag{44}$$

where Δe designates the rotation increment of the manner, equals for example to the antisymmetric part of the gradient of the displacement increment. $(\boldsymbol{e}_3)_o$ designating the vector $\boldsymbol{e}_3$ at the explicitly known instant t; therefore we can perform the correction of this vector given by Eq.(44) prior to any other calculation, without having to perform iterations.

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