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Abstract

Morrey's conjecture deals with two properties of functions which are known as quasi-convexity and rank-one convexity. It is well established that every function satisfying the quasi-convexity property also satisfies rank-one convexity. Morrey (1952) conjectured that the reversed implication will not always hold. In 1992, Šverák found a counterexample to prove that the Morrey's conjecture is true in three dimensional case. The planar case remains, however, open and interesting because of its connections to complex analysis, harmonic analysis, geometric function theory, probability, martingales, differential inclusions and planar non-linear elasticity. Checking analytically these notions is a very difficult task as the quasi-convexity criterion is of non-local type, especially for vector-valued functions. That's why we perform some numerical analysis based on gradient descent algorithms using Dacorogna and Marcellini's [12] example function $f_\gamma(\xi) = \|\xi\|^4 - \gamma\|\xi\|^2 det\xi$ where ξ is a 2×2 matrix. Our numerical results indicate that Morrey's conjecture holds true.

Keywords: Morrey's Conjecture; Quasi-convexity; Gradient Descent; Non-convex Optimization; Iwaniec Conjecture

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1 Introduction

In the 1950's, Charles Morrey worked to find what is the correct notion of convexity in the context of calculus of variations. To address this, he considered the problem of minimizing the functional

$$I(\phi) = \int_\Omega f(\nabla\phi(\mathbf{x}))d\mathbf{x} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is a map, $\nabla\phi \in \mathbb{R}^{n\times m}$ (the set of $n\times m$ matrices), and $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ is a continuous function, together with prescribed Dirichlet conditions on the boundary $\partial\Omega$. This is equivalent to proving that the functional I in Eq.(1.1) is weakly lower semi-continuous in some Sobolev space $\mathbf{W}^{1,p}$. This problem is a difficult one and has not yet received a fully satisfactory answer. It was first formulated by Bliss in 1937 in his seminar on the calculus of variations and has received considerable attention, in particular by Albert [1], Reid [35]. In addition, MacShane [22], Hestenes and MacShane [17], Terpstra [40], Van Hove [41], Serre [37] and Marcellini [23] for the quadratic case and in a more general context by Morrey [27], [28] (see [4], [5], [6], [11] for more details on the quadratic cases). C. Morrey wanted to define the conditions of convexity on the function f (not including growth or smoothness conditions) that guaranteed that the problem Eq.(1.1) is very well posed variational problem.

Morrey proved that the functional I in Eq.(1.1) is weakly lower semi-continuous (ellipticity condition) if and only if the function f is quasi-convex. However, due to its non-local character [21], quasi-convexity of a function is rather difficult to check. This has motivated Morrey to look for local conditions on f that warranted the weakly lower semi-continuity of the integral. It is well known that if the function f is convex then it is also poly-convex [4] which plays a key role in non-linear elasticity. The notion of poly-convexity was introduced in the context of non-linear elasticity theory by Sir John Ball in a pioneering paper [4]. A self contained study giving necessary and sufficient conditions for poly-convexity in arbitrary special dimension was given by Alexander Mielke [25].

More generally, it was established that if a function f defined on $\mathbb{R}^{n\times m}$ is poly-convex, then f is quasi-convex, which also implies that the function f is rank-one convex, a local convexity property. The converse implications do not always hold true; for instance, rank-one convexity does not imply poly-convexity for dimension $n \geq 2$ [12], a sufficient condition for quasi-convexity requiring that the function can be written as a function its minors; references about the latter statement can also be found in [20], [29] and [32], and the references therein. Also, rank-one convexity does not imply quasi-convexity for $n > 2$ [13]. A recent remarkable counterexample, in favor of rank-one convexity does not imply quasi-convexity in general, was introduced by Vladimir Šverák [38] and is valid for $m \geq 3$. Whether this latter implication holds for $n = 2$ is still an open question: the conjecture that rank-one convexity and quasi-convexity are not equivalent is also called Morrey's conjecture, see Morrey [27]. If Morrey's conjecture holds true, many mathematical and/or mechanical modeling methods for material behaviors would have a much more robust theoretical foundations; indeed, for composite materials for instance, the question of whether composites can be constructed with sequential laminates, see [26], would have been resolved as the existence of non quasi-convex but rank-one convex functions is linked to this question. Also, due to their good mathematical structure in terms of variational principles as explained in Gutierrez and Villavicencio [16], quasi-convex functions are used in modeling phase transition in solids as demonstrated in [9], shape optimization (see for instance Pedregal [33]) , and in fracture mechanics of materials, see Francfort and Marigo [14].

For some classes of functions on $\mathbb{R}^{2\times 2}$, however, several works have established that the two convexity properties are equivalent, see for instance the works by [29], [36], [37], [40]. In this spirit, Martin et al. [24] have shown in the context of non-linear elasticity that any energy function $W : GL^{+}(2) \to \mathbb{R}$ which is isotropic and objective (i.e. bi-$SO(2)$-invariant) as well as isochoric is rank-one convex if and only if the energy density W is poly-convex and doing so gives a negative response to Morrey's conjecture as poly-convexity implies quasi-convexity. In addition, for quadratic types of functions, the equivalence between rank-one and quasi-convexity can be established using Plancherel's formula. The resolution of this equivalence could have big impacts in the theory of conformal mappings for two-component maps case. In particular, if the equivalence between rank-one convexity and quasi-convexity for two component-mappings turns out to be true, then the norm of the corresponding Beurling-Ahlfors transform equals $p^{\star} - 1$, see for instance [19]. It is interesting to point out that Morrey's conjecture also has some connections with the Iwaniec conjecture [18], the solution of which could impact the resolution of the Morrey's conjecture problem as we shall later explain in Section 2. Indeed, if the equivalence rank-one convexity and quasi-convexity is true, this would imply that the Iwaniec conjecture is true. By Mathematical contraposition, if the Iwaniec conjecture does not hold true, then rank-one convexity would not imply quasi-convexity. Thus, Morrey's conjecture would have been true.

Numerous attempts have been made to construct examples of functions that are rank-one convex, but not quasi-convex [2], [11], [13]. The complexity of the involved calculations has not yet permitted a complete analytical study of such examples, see for instance [10], [13], [15]. Even, the counterexample provided in Vladmir Šverák [38] seems to be a purely three dimension case as many attempts to translate it into a two dimensional setting failed, see [7], [32], [34] for references. We could not find other counterexamples in the literature. In a recent note, Pedregal [31] provides some evidence in favor of the Morrey's conjecture for two-component maps in dimension two by giving an explicit family of maps parameterized by τ and proving that for small values of τ these maps can not be achieved by lamination. As well stated in [30], this will be equivalent to the assertion that there are some rank-one functions that are not quasi-convex, and thereby confirming the validation of Morrey's conjecture. Even though this approach might yield potential counterexample candidates, Pedregal [31] recognized himself that the procedure tends to be more involved than in the situation examined by Šverák. This is a good reason, as a first step, to use numerical analysis to study the problem of whether or not the Morrey's conjecture is valid. One of such works was performed some years ago by

Dacorogna et al. [13] on the example of Dacorogna and Marcellini [12] energy density function; the numerical results of these investigations indicated that Dacorogna and Marcellini example which is rank-one convex, is also quasi-convex.

The problem considered in [13] is as follows: for $\xi \in \mathbb{R}^{2\times 2}$ and $\phi \in \mathbf{W}_0^{1,4}(\Omega;\mathbb{R}^2)$, Dacorogna et al. let

$$f_\gamma(\xi) = \|\xi\|^4 - \gamma\|\xi\|^2 det\xi, \tag{1.2}$$

and

$$J_\gamma(\xi,\phi) = \int_\Omega [f_\gamma(\xi + \nabla\phi(x)) - f_\gamma(\xi)]dx. \tag{1.3}$$

They choose $\Omega = (0,1)\times(0,1)$ and found that the quasi-convexity of f_γ is then equivalent to

$$\inf_{\xi\in\mathbb{R}^{2\times 2}} \inf_{\phi\in\mathbf{W}_0^{1,4}(\Omega;\mathbb{R}^2)} \{J_\gamma(\xi,\phi)\} = 0. \tag{1.4}$$

A few remarks are at hand here:

1. First, note that because of the homogeneity of f_γ, the infimum in Eq.(1.3) is either 0 or $-\infty$.
2. Next, it follows, if $\gamma > \dfrac{4}{\sqrt{3}}$, then f_γ is not rank-one convex and therefore in Eq.(1.3) the infimum is $-\infty$.

Dacorogna et al. [13] described a numerical approximation to this problem by defining a positive integer N and $h = 1/N$, and a partition Ω into $\Omega_{ij} = (ih,(i+1)h)\times(jh,(j+1)h), 0\le i,j\le N-1$. Each of these Ω_{ij} is subdivided into two triangles. They denoted τ_k this triangulation of Ω and the triangles by K and they let $P_1, P_2, \ldots, P_M, M = (N-1)^2$, be the internal nodes. Next, they set

$$\begin{aligned} V_h &= \{u \in C^0(\overline{\Omega}) : u \text{ is affine on each } K \in \tau_h \text{ and } u = 0 \text{ on } \partial\Omega\}, \\ \mathbf{W}_h &= V_h \times V_h \subset \mathbf{W}_0^{1,4}(\Omega;\mathbb{R}^2). \end{aligned}$$

By fixing $\xi \in \mathbb{R}^{2\times 2}$ Dacorogna et al. [13] minimize J_γ over $\mathbf{W}_h$ using a gradient descent method, obtained by defining $w^l, l = 1,2,\ldots,L$, $d^l = \nabla J_\gamma(w^l)$, $g^l(\alpha) = J_\gamma(w^l + \alpha d^l)$, and updating w^l using the explicit gradient update $w^{l+1} = w^l + \overline{\alpha} d^l$, where $\overline{\alpha}$ is obtained by solving $\frac{dg^l}{d\alpha} = 0$, using only one step in Newton's method with starting point $\alpha = 0$.

The numerical approach used in [13] to solve the above problem is based on a steepest descent algorithm with a crude approximation on the derivation of the gradient of the functional to be minimized. Gremaud [15] used a different numerical approach for the same problem. Unlike in [13], the corresponding minimizing problem was solved using an annealing-like algorithm. Gremaud's results [15] showed that the example functions considered in [12] are quasi-convex if and only if they are rank-one convex, contradicting Morrey's conjecture, but confirming Iwaniec conjecture.

Other numerical computation strategies to assess Dacorogna and Marcellini [12] examples functions with respect to its abilities to provide insights onto the validation/invalidation of Morrey's conjecture exist. Recent works by the authors from Duke University [10] improved upon the numerical simulations of Dacorogna in an attempt to define a function that is rank-one convex, but not quasi-convex. Duke University's simulations improved on the computational speed and the numerical optimization techniques since the publication of Dacorogna's works. We want to report here also that there has been several numerical attempts to address Morrey's conjecture problem outside of the context of the example proposed by [12]; among these, let us mention the work of Gutierrez and Villenvicencio [16] where the authors derived an optimization algorithm based on (i) some necessary condition for the quasi-convexity of fourth-degree polynomials that distinguishes between quasi-convex and rank-one convex functions in the three dimensional case, (ii) a characterization of rank-one convex fourth-degree polynomials in terms of infinitely many constraints.

The objective of this report is to go somewhat beyond the pioneering works of Dacorogna et al. [13]. We do this by improving on the numerical algorithm these authors used in the gradient descent strategy they proposed. Namely, we calculate the exact expression of the gradient of the functional involved in the optimization problem at hand here and used their approximated values. We solved the minimization problem numerically by using the approximated values of the of the trial functions ϕ at each of the nodes of the mesh we used to model the domain Ω. Note here that these values are obtained from an initial trial function ϕ that we choose as oscillating functions since in Dacorogna et al.'s numerical computations, these types of functions seem to be promising. Once the updated values of the trial functions at the nodes are obtained, we used them to check the Jensen's inequality which associated to the quasi-convexity property of the function f_γ. The initial trial functions to enter the steepest descent iterative algorithm are chosen together with some fixed value of the matrix ξ. By randomizing the entries of ξ, we successfully used for each of the iteration a new matrix ξ. The results indicate that for an appropriate choice of γ in the function f_γ and ξ in Dacorogna [13], f_γ is rank-one convex, but the Jensen inequality defining the quasi-convexity is violated, and thereby, for these values of γ and ξ, f_γ is rank-one convex, but is not quasi-convex, thus confirming that, at least from the numerical stand point, the Morrey's conjecture holds true. The report is organized as follows:

- In Section 2, we present the importance of Morrey's conjecture from the point of views of calculus of variations, harmonic analysis and some connections of Morrey's conjecture with that of Iwaniec, and finally, from the differential inclusion perspective.
- Next, in Section 3, we describe the problem to be solved numerically. Note that a brief overview of this description was presented above in this introduction.
- The following Sections 4.2 and 4.3 present Gremaud's [15] and Duke University's [10] approaches and results to solve the problem under consideration and which we presented in Section 3.
- In Section 5, we demonstrate the numerical implementation methods we used as well as the results we obtained.

- Furthermore, in Section 5.3, we present an improvement of Duke University's approach [10] by refining the oscillating functions involved in checking Jensen inequality and defining a gradient descent algorithm using the trial functions Duke University has proposed. Our results improved on the ones from Duke University [10].
- Finally, Section 6 discusses the results we obtained from all the numerical simulations.

2 Importance of Morrey's conjecture

2.1 Calculus of Variations

Let $\Omega \subset \mathbb{R}^2$ be a bounded of C^1 boundary. We fix $r \in (1, \infty)$, and endow $W^{1,r}(\Omega, \mathbb{R}^2)$ with the weak topology.
(i) Let $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be such that there exists and $C_0, C_1 > 0$ such that

$$-C_0 + \frac{1}{C_1}\|\xi\|^r \leq F(\xi) \leq C_1(\|\xi\|^r + 1) \qquad \forall \xi \in \mathbb{R}^{2\times 2}.$$

Define

$$I(u) = \int_\Omega F(Du)dx.$$

The necessary and sufficient condition for I to be weakly lower semi-continuous is that F is quasi-convex. These statements mean that the sub-level sets on I are weakly compact if and only if F is quasi-convex. In other words, I to admit a minimizer on every weakly compact set $\mathcal{K} \subset W^{1,r}(\Omega, \mathbb{R}^2)$ if and only if F is quasi-convex. Such a result can be extended to functions $F \equiv F(x, u, \cdot)$, where $x \in \Omega$, $u \in \mathbb{R}^2$ to encompass functional appearing in non-linear elasticity theory. Therefore, understanding quasiconvexity is central in applying the calculus of variations to non-linear elasticity theory.

2.2 Harmonic analysis

We denote the Fourier transform operator by $\hat{\mathcal{F}}$ and the complex conjugate of $z \in$ by $\bar{z}$. We consider the singular kernel $m : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ defined by $m(z) = \bar{z}/z$ and the operators D_z and $D_{\bar{z}}$ defined on the set of smooth functions on the complex plane by

$$D_z u = \frac{1}{2}\Big(\partial_x u - i\partial_y u\Big), \quad D_{\bar{z}} u = \frac{1}{2}\Big(\partial_x u + i\partial_y u\Big).$$

The Beurling-Ahlfors operator $B : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ if defined for $u : \mathbb{C} \to \mathbb{C}$ by

$$Bu(z) = -\frac{1}{\pi}\text{p.v.} \int_{\mathbb{C}} \frac{u(w)}{(z-w)^2} dw$$

When $p = 2$ $\|B\|_{L^2} = 1$ since T can be obtained by the formula

$$\hat{\mathcal{F}}\Big(Bu\Big)(z) = (m\hat{\mathcal{F}}u)(z).$$

In harmonic analysis, this operator plays an important role since it satisfies the property

$$B(D_{\bar{z}}u) = D_z u$$

when u is a smooth function on the complex plane. An outstanding open problem of the past decades is the computation of the L^p norm of B for $1 < p < \infty$.

We identify $\mathbb{C}\times\mathbb{C}$ with $\mathbb{R}^{2\times 2}$ via the map $\mathbb{C}\times\mathbb{C}\to\mathbb{R}^{2\times 2}$ given by

$$T(z,w)=\begin{pmatrix}\mathrm{Re}(z)+\mathrm{Re}(w) & \mathrm{Im}(z)-\mathrm{Im}(w)\\ \\ \mathrm{Im}(z)+\mathrm{Im}(w) & \mathrm{Re}(w)-\mathrm{Re}(z)\end{pmatrix}\qquad \forall z,w\in\mathbb{C}.$$

We can convert any function $U:\mathbb{C}\times\mathbb{C}\to\mathbb{R}$ to a function $U^{\#}:\mathbb{R}^{2\times 2}\to\mathbb{R}$ through the identity

$$U^{\#}\big(T(z,w)\big)=-U(z,w)\qquad \forall z,w\in\mathbb{C}.$$

In particular, one checks that the function

$$U(z,w):=p\Big(1-\frac{1}{p^*}\Big)^{p-1}\Big(|w|-(p^*-1)|z|\Big)\big(|z|+|w|\big)^{p-1},\qquad \forall z,w\in\mathbb{C}.$$

is such that for all $z,w,h,k\in\mathbb{C}$ such that $|k|\le|h|$, $t\to U(z+th,w+tk)$ is concave on the real line. The relation

$$U(z+th,w+tk)=-U^{\#}\big(T(z,w)+th'\otimes k'\big),\qquad \forall k',h'\in\mathbb{R}^2,$$

where

$$h=\Big(h_1'k_1'-h_2'k_2'\Big)+i\Big(h_1'k_2'+k_1'h_2'\Big),\quad k=\Big(h_1'k_1'+h_2'k_2'\Big)+i\Big(k_1'h_2'-h_1'k_2'\Big)$$

allows to conclude that $U^{\#}$ is rank-one convex since $|h|=|k|$.

An open question in harmonic analysis is to know if

$$\int_{\mathbb{C}}U\Big(D_{\bar z}u,D_zu\Big)dA\le 0,\qquad \forall u\in C_0^{\infty}(\mathbb{C}). \tag{2.1}$$

Since $U^{\#}$ is rank-one convex, if Morrey's conjecture was false, we would conclude that $U^{\#}$ is quasi-convex. In particular, we would have

$$0=U^{\#}(0)\le\int_{\mathrm{spt}\phi}U^{\#}(D\phi)=-\int_{\mathrm{spt}\phi}U\Big(D_{\bar z}f,D_zf\Big)dA. \tag{2.2}$$

where $\phi=(\mathrm{Ref},\mathrm{Imf})$. One can show that $U^{\#}$ quasiconvex, implies Iwaniec conjecture which asserts that

$$\|B\|_{L^p}=\max\left\{p,\frac{p}{p-1}\right\}-1.$$

In case $U^{\#}$ is not quasi-convex, we would have resolved Morrey's conjecture. If Iwaniec conjecture is false then Morrey's conjecture is true. It is believed by experts [39] that $U^{\#}$ is quasi-convex and so, Iwaniec conjecture would be true. Therefore, a negative or positive answer to Iwaniec conjecture is of first importance. We plan to investigate the connections between the Morrey and Iwaniec conjectures in a near future work.

3 Definitions and Preliminaries

In order to describe our methodologies and results we need the following definitions.

1. f is said to be *quasi-convex* if

$$\int_{\Omega} f(\xi + \nabla\phi(x))dx \geq f(\xi) \text{ meas } \Omega \tag{3.1}$$

   for every $\xi \in \mathbb{R}^{2\times 2}$ and for every $\phi \in \mathbf{W}_0^{1,\infty}(\Omega;\mathbb{R}^2)$ (the set of Lipschitz functions vanishing on $\partial\Omega$).

2. f is said to be *rank-one convex* if

$$f(\lambda\xi + (1-\lambda)\eta) \leq \lambda f(\xi) + (1-\lambda)f(\eta) \tag{3.2}$$

   for every $\lambda\xi[0,1], \xi, \eta \in \mathbb{R}^{2\times 2}$ with $det(\xi - \eta) = 0$ (if $\xi = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$, then $det\xi = \xi_{11}\xi_{22} - \xi_{12}\xi_{21}$).

3. f is said to be *poly-convex* if there exists $\phi : \mathbb{R}^5 \to \mathbb{R}$ convex such that

$$f(\xi) = \phi(\xi, det\xi) \tag{3.3}$$

   for every $\xi \in \mathbb{R}^{2\times 2}$.

Some observations can be made here:

1. It can be proved (see [11] for example) that if Eq.(3.1) holds for one domain Ω, it holds for any domain.

2. In the second definition we can see that if f is C^2, then the rank-one convexity of f is equivalent to the classical *Legendre-Hadamard* condition

$$\sum_{i,j=1}^{2} \sum_{\alpha,\beta=1}^{2} \frac{\partial^2 f(\xi)}{\partial\xi_{i\alpha}\partial\xi_{j\beta}} \lambda_i\lambda_j\mu_\alpha\mu_\beta \geq 0 \tag{3.4}$$

   for every $\xi \in \mathbb{R}^{2\times 2}$ and every $\lambda, \mu \in \mathbb{R}^2$.

In general, one has the following diagram:

$$\begin{array}{c} f \text{ convex} \underset{\Longrightarrow}{\not\Longleftarrow} f \text{ poly-convex} \underset{\Longrightarrow}{\not\Longleftarrow} f \text{ quasi-convex} \underset{\Longrightarrow}{\overset{?}{\Longleftarrow}} f \text{ rank-one convex} \\ \Updownarrow \\ I \text{ is weakly lower} \\ \text{semi-continuous(w.l.s.c.)} \end{array}$$

Morrey's Conjecture. It was conjectured by Morrey [27], that in fact f rank-one convex $\not\Longrightarrow$ f quasi-convex.

It is the aim of this report to study the rank-one convexity of some functions f and their quasi-convexity and marginally their poly-convexity. The study of the connection between rank-one convexity and poly-convexity was done by Dacorogna and Marcellini [12] and Voss et al.[42], but we have not yet explored such features because our main objective is to illustrate the validity of Morrey's conjecture. Our main example function is the example in Dacorogna and Marcellini [12]

$$f_\gamma(\xi) = \|\xi\|^2 \left(\|\xi\|^2 - \gamma \, det\xi\right). \tag{3.5}$$

Because of the above observation, and in view of the above diagram, f is a potential candidate for answering the Morrey's conjecture. Other candidate functions exist, see for instance [42] in the context of the description of the mapping used in the kinematics involved in developing constitutive relations governing the mechanical behavior of solids or fluids. However, analytical computations seem to be a very hard method for deciding whether or not the function f is quasi-convex. This is due to the non-local nature of the quasi-convexity criterion.

4 Previous Efforts on the Morrey's Conjecture Based on Dacorogna and Marcellini's Example Function

4.1 Dacorogna and Marcellini's Example Function

In 1990, Dacorogna et al.[13] investigated the example function he introduced with Marcellini[12] in 1988. The function is of the form

$$f_\gamma(\xi) = \|\xi\|^2 \left(\|\xi\|^2 - \gamma \, det\xi\right) \tag{4.1}$$

where $\xi \in \mathbb{R}^{2\times 2}$, and the norm is the usual Euclidean norm.

γ plays a key role here and Dacorogna et al.[13] successfully proved that for $2 < \gamma \leq \dfrac{4}{\sqrt{3}}$, f is rank-one convex but not poly-convex. Recall the general implication diagram:

$$f_\gamma \;\; \text{poly-convex} \implies f_\gamma \;\; \text{quasi-convex} \implies f_\gamma \;\; \text{rank-one convex} \tag{4.2}$$

Dacorogna et al.[13] then deduced that such function $f_\gamma(\xi)$ with $2 < \gamma \leq \dfrac{4}{\sqrt{3}}$ was a good candidate to explore Morrey's conjecture that

$$f_\gamma \text{ rank-one convex } \;\not\Longrightarrow\; f_\gamma \text{ quasi-convex}$$

Dacorogna et al.[13] defined a function J as follows:

$$J_\gamma(\xi, \phi) = \int_\Omega [f_\gamma(\xi + \phi(x)) - f_\gamma(\xi)]dx \tag{4.3}$$

then stated that the quasiconvexity of f is equivalent to

$$\inf_{\xi \in \mathbb{R}^{2\times 2}} \inf_{\phi \in W_0^{1,4}(\Omega;\mathbb{R}^2)} \{J_\gamma(\xi, \phi)\} = 0 \tag{4.4}$$

Realizing the difficulty in determining a global property like quasiconvexity, Dacorogna et al.[13] therefore transformed this problem into a optimization problem and performed steepest descent algorithm to minimize J_γ with different choices of ξ and ϕ. Dacorogna et al.[13] concluded that their numerical results shown below tended to indicate that function f is quasi-convex if and only if f is rank-one convex, leaving Morrey's conjecture unanswered:

1. With random ξ and ϕ, the closest γ to $\dfrac{4}{\sqrt{3}}$ that made $J_\gamma(\phi)$ approaches 0 is 2.31.
2. With $\xi = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$, the closest γ to $\dfrac{4}{\sqrt{3}}$ that made $J_\gamma(\phi)$ approaches 0 is 2.31.
3. With $\xi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, the closest γ to $\dfrac{4}{\sqrt{3}}$ that made $J_\gamma(\phi)$ approaches 0 is 3.25.

4.2 Gremaud's Simulated Annealing Method and Results

In 1993, another mathematician named Gremaud[15] studied the same function f. Similar to Dacorogna et al.[13], Gremaud also treated this problem as a numerical optimization problem. He used simulated annealing method which is described below. His method incorportaed some stochastic analysis techniques which were known to have the benefit of not getting stuck at local minimums[15]:

1: Choose $\gamma \geq \dfrac{4}{\sqrt{3}}$
2: Initialize ϕ^0, $k = 0$
3: Calculate $g(0) = \nabla J(\phi^0)$
4: Let $\upsilon(t) = \dfrac{c}{log(t+2)}$ with $c > 0$
5: Let $g^k = \nabla J(\phi^k) + \beta^k g^{k-1}$, where $\beta^k = \frac{\left(\nabla J(\phi^k) - \nabla J(\phi^{k-1}), \nabla J(\phi^k)\right)}{\|\nabla J(\phi^{k-1})\|^2}$
6: Let $t^k = \sum_{i=0}^{n-1} \tau^i$, where τ^k is experimentally determined
7: Let $\tilde{w}^k = w(t^{k+1}) - w(t^k)$ is a randomly generated vector in $\mathbb{R}^2$ such that $\tilde{w}^k \in [-1, 1]^2$
8: $\phi^{k+1} = \phi^k - \tau^k g^k + \sqrt{2\upsilon(t^k)}\tilde{w}^k$
9: **if** $J(\phi^{k+1}) \ll 0$ (non quasi-convexity) **then**
10: Set $\gamma \Leftarrow \gamma - \delta\gamma$, $\phi^0 \Leftarrow \phi^{k+1}$, $k = 0$
11: Go to step 3
12: **else**
13: Set $k \Leftarrow k + 1$
14: Go to step 3 (Until reach max iterations)
15: **end if**

We present his results below in Figure 1. Note that these γ values are almost $\frac{1}{2}$ of what we saw in Dacorogna's results before. This is because the function f Gremaud used was in the form of

$$f_\gamma(\xi) = \|\xi\|^4 - 2\ \|\xi\|^2\ \gamma\ det\xi. \tag{4.5}$$

As we can see from Figure 1, when the value of 2γ gets closer to $\dfrac{4}{\sqrt{3}}$, the time it takes for J to become negative goes to infinity. Gramaud speculated that his results were a sign of

$$\lim_{2\gamma \to \frac{4}{\sqrt{3}}} \left\{ \inf_{\xi \in \mathbb{R}^{2\times 2}} \inf_{\phi \in W_0^{1,4}(\Omega;\mathbb{R}^2)} \{J_\gamma(\xi, \phi)\} \right\} = 0 \tag{4.6}$$

which, according to Dacorogna et al. [13], is equivalent to the quasiconvexity of f. Gremaud [15] concluded that this observation suggested function f is quasi-convex if and only if it is rank-one convex.

4.3 Duke University's Improvement of Dacorogna's Approach

Recently, a group of researchers from the Duke University [10] attempted to solve Morrey's Conjecture using the function f Dacorogna and Marcellini [12]

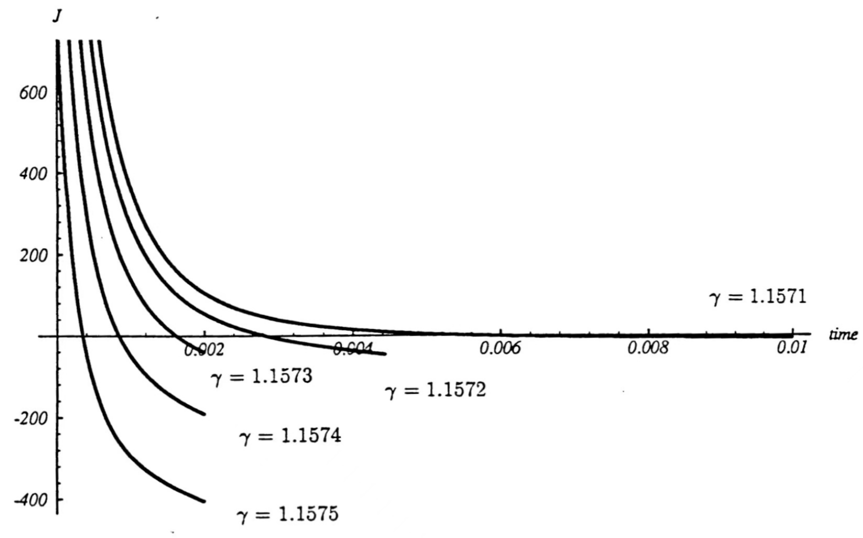

Figure 1: Gremaud: Evolution of J for various values of γ

introduced. Rather than doing numerical optimization on $J_\gamma(\xi)$ like Dacorogna and Gremaud, they turned their attention to γ. They began by expanding and rearranging $J_\gamma(\xi)$, then solving for γ:

$$\gamma^* = \sup_{\phi \in W_0^{1,\infty}(\Omega,\mathbb{R})} \{ \frac{\int_\Omega \|\xi + \nabla\phi\|^4 - \|\xi\|^4 d\Omega}{\int_\Omega \|\xi + \nabla\phi\|^2 det(\xi + \nabla\phi) - \|\xi\|^2 det\xi d\Omega} \} \tag{4.7}$$

Then with the two different ξ values advised by Dacorogna, they got:

1) For $\xi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, the new constraint for γ:

$$\gamma > \frac{\int_\Omega \|\nabla\phi\|^4}{\int_\Omega \|\nabla\phi\|^2 det(\nabla\phi)} \tag{4.8}$$

They tried different constructions of the two-component function ϕ and minimized the ratio in the above inequality. Their choices of ϕ are all seperable functions, and one example is in the form of

$$\phi = \begin{bmatrix} g(x)h(y) \\ u(x)v(y) \end{bmatrix} \tag{4.9}$$

where

$$g(x) = a_0 + \sum_{i=1}^{n} a_i sin(ix) + \sum_{j=1}^{n} b_j cos(jx), \tag{4.10}$$

and functions h, u, and v are all of similar separable forms.

Due to the improvements in computational speed and optimization techniques, also due to their smart choice of the function ϕ, they were able to improve upon the numerical results of Dacorogna. With some reasonable n, the researchers were able to derive $\gamma = 3.19$, surpassing Dacorogna's result $\gamma = 3.25$. However, all their results were still not close enough to $\frac{4}{\sqrt{3}}$.

2) For $\xi = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$, they replaced ξ in the definition of quasi-convexity by $\lambda\alpha + (1-\lambda)\beta$ and got:

$$\begin{aligned} \int_\Omega f(\lambda\alpha + (1-\lambda)\beta + \nabla\phi(x)) &= \lambda f(\alpha) + (1-\lambda)f(\beta) + O(\sqrt{\epsilon}) \\ &\leq f(\lambda\alpha + (1-\lambda)\beta) \end{aligned} \tag{4.11}$$

where the last line holds from the quasi-convexity of f. As $\epsilon \to 0$, f will also become rank-one convex. From this idea, the criterion for non quasi-convexity is equivalent to finding an $\epsilon > 0$ and $\delta \in \mathbb{R}^{2\times 2}$ such that

$$\frac{1}{2} f_\gamma(\xi + \delta) + \frac{1}{2} f_\gamma(\xi - \delta) - f_\gamma(\xi) + O(\sqrt{\epsilon}) < 0 \tag{4.12}$$

Clearly, this can be achieved by choosing δ and ϵ close enough to 0, and different δ and ϵ values were found with some fixed γ. With $\xi = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$, the closest γ Dacorogna obtained in his research was 2.31, which can be written as $\frac{4}{\sqrt{3}} + O(10^{-2})$. The researchers instead set γ as $\frac{4}{\sqrt{3}} + O(10^{-4})$, and found that with $\epsilon = 10^{-4}$ and $\delta = \begin{bmatrix} 10^{-3} & 0 \\ 0 & 0 \end{bmatrix}$, the inequality in 2.11 holds which means f is not quasi-convex, improving on Dacorogna's result and getting closer to $\frac{4}{\sqrt{3}}$.

5 UCLA's Methodologies

5.1 Steepest Descent Algorithm

5.1.1 Calculation of Gradient of the Functional

In this research, we will restrict our attention to the function introduced by Dacorogna and Marcellini [12], i.e.,

$$f_\gamma(\xi) = \|\xi\|^2 \left(\|\xi\|^2 - \gamma\, det\xi\right). \tag{5.1}$$

Our purpose of this algorithm is to find some suitable ϕ, ξ, and γ so we can construct a counterexample which is rank-one convex but not quasi-convex. Chances will be higher if we can generate and cover different permutations of ϕ, ξ, and γ. To do the task, we perform a steepest descent algorithm described below on the function ϕ.

Recall that $\phi \in W_0^{1,4}(\Omega; \mathbb{R}^2)$, which is a set of Lipshitz functions that vanish on $\partial\Omega$, we write

$$\phi := \begin{bmatrix} \phi_1(x_1, x_2) \\ \phi_2(x_1, x_2) \end{bmatrix} \tag{5.2}$$

It is also worth marking out notations that

$$\xi = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix}, \tag{5.3}$$

$$J_\gamma(\phi) = \int_\Omega f_\gamma(\xi + \nabla\phi) dx. \tag{5.4}$$

The steepest descent algorithm we use is in the form:

$$\phi^{k+1} = \phi^k - \tau \delta J_\gamma(\phi^k) \tag{5.5}$$

where τ, the step size, is determined in each iteration.

What's left to do is solving for $\delta J_\gamma(\phi^k)$. We notice the difficulty and complexity in finding this gradient of the functional J_γ directly, so we proceed from writing out the gateaux derivative of J_γ first.

The gateaux differential $dJ(\phi; \psi)$ of J_γ at $\phi \in \Omega$ in the direction of $\psi \in \mathbb{R}^2$ can be approached by

$$\begin{aligned} J_\gamma(\phi + \varepsilon\psi) &= \int_\Omega f_\gamma(\xi + \nabla\phi + \varepsilon\nabla\psi) - f_\gamma(\xi + \nabla\phi) dx \\ &= \varepsilon \int_\Omega \sum_{i,j=1}^{2} \frac{\partial}{\partial \xi_{ij}} (\xi + \nabla\phi) \frac{\partial \psi_i}{\partial x_j} dx + o(\varepsilon) + J_\gamma(\phi). \end{aligned} \tag{5.6}$$

Subtracting $J_\gamma(\phi)$ from both sides, the above expression, as we send ϵ to 0, yields $dJ(\phi;\psi)$

$$\begin{aligned} dJ(\phi;\psi) &= \lim_{\varepsilon\to 0} \frac{J_\gamma(\phi+\varepsilon\psi) - J_\gamma(\phi)}{\varepsilon} \\ &= \sum_{i,j=1}^{2} \int_\Omega \frac{\partial f_\gamma}{\partial \xi_{ij}}(\xi+\nabla\phi)\frac{\partial \psi_i}{\partial x_j}dx \\ &= -\sum_{i,j=1}^{2} \int_\Omega \frac{\partial}{\partial x_j}\frac{\partial}{\partial \xi_{ij}} f_\gamma(\xi+\nabla\phi)\psi_i \; dx \end{aligned} \tag{5.7}$$

which, by the definition of gateaux derivative, is also

$$dJ(\phi;\psi) = \langle \delta J_\gamma(\phi), \psi\rangle := \int_\Omega \delta J_\gamma(\phi(x))\psi(x)dx. \tag{5.8}$$

From Eq.(5.7) and Eq.(5.8), it follows easily that

$$\delta J_\gamma(\phi) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\frac{\partial}{\partial \xi_{ij}} f_\gamma(\xi+\nabla\phi). \tag{5.9}$$

Using the result in Eq.(5.9), our algorithm can now be expressed as

$$\phi^{k+1} = \phi^k + \tau \sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\frac{\partial}{\partial \xi_{ij}} f_\gamma(\xi+\nabla\phi^k). \tag{5.10}$$

Since $\phi \in \mathbb{R}^2$, we perform gradient descent separately on each of the components

$$\begin{cases} \phi_1^{k+1} & = \phi_1^k + \tau \sum_{j=1}^2 \frac{\partial}{\partial x_j}\frac{\partial}{\partial \xi_{1j}} f_\gamma(\xi+\nabla\phi^k) \\ & = \phi_1^k + \tau(\frac{\partial}{\partial x_1}\frac{\partial}{\partial \xi_{11}} f_\gamma(\xi+\nabla\phi^k) + \frac{\partial}{\partial x_2}\frac{\partial}{\partial \xi_{12}} f_\gamma(\xi+\nabla\phi^k)) \\ \phi_2^{k+1} & = \phi_2^k + \tau \sum_{j=1}^2 \frac{\partial}{\partial x_j}\frac{\partial}{\partial \xi_{1j}} f_\gamma(\xi+\nabla\phi^k) \\ & = \phi_2^k + \tau(\frac{\partial}{\partial x_1}\frac{\partial}{\partial \xi_{12}} f_\gamma(\xi+\nabla\phi^k) + \frac{\partial}{\partial x_2}\frac{\partial}{\partial \xi_{22}} f_\gamma(\xi+\nabla\phi^k)) \end{cases} \tag{5.11}$$

There are four unknown partial derivatives in Eq.(5.11). We will show the complete steps for the first gradient. Since all four calculations are largely identical, we will only present the results of the other three.

Let $\nabla_j\phi_i$ denote $\dfrac{\partial}{\partial x_j}\phi_i$, then

$$\xi + \nabla\phi = \begin{bmatrix} \xi_{11} + \nabla_1\phi_1 & \xi_{12} + \nabla_2\phi_1 \\ \xi_{21} + \nabla_1\phi_2 & \xi_{22} + \nabla_2\phi_2 \end{bmatrix} \tag{5.12}$$

As $\xi + \nabla\phi$ is a regular 2×2 matrix, it is clear that its determinant is

$$\begin{aligned} det(\xi+\nabla\phi) =& (\xi_{11} + \nabla_1\phi_1)(\xi_{22} + \nabla_2\phi_2) \\ & - (\xi_{12} + \nabla_2\phi_1)(\xi_{21} + \nabla_1\phi_2) \end{aligned} \tag{5.13}$$

and its norm is

$$\begin{aligned}\|\xi+\nabla\phi\|^2 =&(\xi_{11}+\nabla_1\phi_1)^2+(\xi_{12}+\nabla_2\phi_1)^2 \\ &+(\xi_{21}+\nabla_1\phi_2)^2+(\xi_{22}+\nabla_2\phi_2)^{\cdot}\end{aligned} \tag{5.14}$$

Let's start by computing

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial \xi_{11}} f_\gamma(\xi+\nabla\phi). \tag{5.15}$$

To this end, we first consider the inner partial derivative

$$\frac{\partial}{\partial \xi_{11}} f_\gamma(\xi+\nabla\phi) \tag{5.16}$$

which we calculate to be

$$\frac{\partial}{\partial \xi_{11}} f_\gamma(\xi+\nabla\phi) = \frac{\partial}{\partial \xi_{11}}\|\xi+\nabla\phi\|^4 - \frac{\partial}{\partial \xi_{11}}(\gamma\ \|\xi+\nabla\phi\|^2\ det(\xi+\nabla\phi)). \tag{5.17}$$

To make things cleaner, we calculate one term at a time,

$$\begin{aligned}\frac{\partial}{\partial \xi_{11}}\|\xi+\nabla\phi\|^4 &= \frac{\partial}{\partial \xi_{11}}(\|\xi+\nabla\phi\|^2)^2 \\ &= 2\|\xi+\nabla\phi\|^2\frac{\partial}{\partial \xi_{11}}(\|\xi+\nabla\phi\|^2)\end{aligned} \tag{5.18}$$

where

$$\frac{\partial}{\partial \xi_{11}}(\|\xi+\nabla\phi\|^2) = 2(\xi_{11}+\nabla_1\phi_1). \tag{5.19}$$

From Eq.(5.18) and Eq.(5.19),

$$\begin{aligned}\frac{\partial}{\partial \xi_{11}}\|\xi+\nabla\phi\|^4 &= 2\|\xi+\nabla\phi\|^2(2(\xi_{11}+\nabla_1\phi_1)) \\ &= 4(\xi_{11}+\nabla_1\phi_1)\|\xi+\nabla\phi\|^2.\end{aligned} \tag{5.20}$$

Now by the chain rule, we have

$$\begin{aligned}\frac{\partial}{\partial \xi_{11}}(\gamma\ \|\xi+\nabla\phi\|^2\ det(\xi+\nabla\phi)) &= \gamma\frac{\partial}{\partial \xi_{11}}(\|\xi+\nabla\phi\|^2\ det(\xi+\nabla\phi)) \\ &= \gamma\frac{\partial}{\partial \xi_{11}}(\|\xi+\nabla\phi\|^2)det(\xi+\nabla\phi) \\ &\quad+\gamma\|\xi+\nabla\phi\|^2\frac{\partial}{\partial \xi_{11}}det(\xi+\nabla\phi)\end{aligned} \tag{5.21}$$

where

$$\gamma\frac{\partial}{\partial \xi_{11}}(\|\xi+\nabla\phi\|^2)det(\xi+\nabla\phi) = 2\ \gamma\ (\xi_{11}+\nabla_1\phi_1)\ det(\xi+\nabla\phi) \tag{5.22}$$

and

$$\gamma\|\xi+\nabla\phi\|^2\frac{\partial}{\partial \xi_{11}}det(\xi+\nabla\phi) = \gamma\ (\xi_{22}+\nabla_2\phi_2)\ \|\xi+\nabla\phi\|^2. \tag{5.23}$$

Combining the two equations Eq.(5.22) and Eq.(5.23), we can obtain

$$\begin{aligned}\frac{\partial}{\partial \xi_{11}}(\gamma\ \|\xi+\nabla\phi\|^2\ det(\xi+\nabla\phi)) = 2\ \gamma\ (\xi_{11}+\nabla_1\phi_1)\ det(\xi+\nabla\phi) \\ +\ \gamma\ (\xi_{22}+\nabla_2\phi_2)\ \|\xi+\nabla\phi\|^2.\end{aligned} \tag{5.24}$$

Now using the results from Eq.(5.20) and Eq.(5.24), we have

$$\begin{aligned}\frac{\partial}{\partial \xi_{11}} f_\gamma(\xi+\nabla\phi) =& 4(\xi_{11}+\nabla_1\phi_1)\|\xi+\nabla\phi\|^2 - (2\ \gamma\ (\xi_{11}+\nabla_1\phi_1) \\ & det(\xi+\nabla\phi) + \gamma\ (\xi_{22}+\nabla_2\phi_2)\ \|\xi+\nabla\phi\|^2).\end{aligned} \tag{5.25}$$

The remaining task is to find the partial derivative of Eq.(5.25) with respect to x_1:

$$\begin{aligned}\frac{\partial}{\partial x_1}\frac{\partial}{\partial \xi_{11}} f_\gamma(\xi+\nabla\phi) =& \frac{\partial}{\partial x_1}(4(\xi_{11}+\nabla_1\phi_1)\|\xi+\nabla\phi\|^2) \\ & -(\frac{\partial}{\partial x_1}(2\ \gamma\ (\xi_{11}+\nabla_1\phi_1)det(\xi+\nabla\phi)) \\ & +\frac{\partial}{\partial x_1}(\gamma\ (\xi_{22}+\nabla_2\phi_2)\ \|\xi+\nabla\phi\|^2))\end{aligned} \tag{5.26}$$

We separate the work as before. First, we have

$$\begin{aligned}\frac{\partial}{\partial x_1}(4(\xi_{11}+\nabla_1\phi_1)\|\xi+\nabla\phi\|^2) =& 4(\frac{\partial}{\partial x_1}(\xi_{11}+\nabla_1\phi_1))\|\xi+\nabla\phi\|^2 \\ & +4(\xi_{11}+\nabla_1\phi_1)(\frac{\partial}{\partial x_1}\|\xi+\nabla\phi\|^2)\end{aligned} \tag{5.27}$$

where the first term

$$4(\frac{\partial}{\partial x_1}(\xi_{11}+\nabla_1\phi_1))\|\xi+\nabla\phi\|^2 = 4\ \nabla_{11}\phi_1\|\xi+\nabla\phi\|^2 \tag{5.28}$$

and we leave the second term unexpanded for now.

For the second term in Eq.(5.26), we obtain

$$\begin{aligned}&\frac{\partial}{\partial x_1} 2\ \gamma\ (\xi_{11}+\nabla_1\phi_1)det(\xi+\nabla\phi) \\ &\qquad = 2\ \gamma\ (\frac{\partial}{\partial x_1}(\xi_{11}+\nabla_1\phi_1))det(\xi+\nabla\phi)) \\ &\qquad\quad +2\ \gamma\ (\xi_{11}+\nabla_1\phi_1)(\frac{\partial}{\partial x_1}det(\xi+\nabla\phi)) \\ &\qquad = 2\ \gamma\ \nabla_{11}\phi_1\ det(\xi+\nabla\phi) \\ &\qquad\quad +2\ \gamma\ (\xi_{11}+\nabla_1\phi_1))(\frac{\partial}{\partial x_1}det(\xi+\nabla\phi))\end{aligned} \tag{5.29}$$

Similarly, for the third term in Eq.(5.26), we derive

$$\begin{aligned}
&\frac{\partial}{\partial x_1}(\gamma\ (\xi_{22} + \nabla_2\phi_2)\ \|\xi + \nabla\phi\|^2)) \\
&\qquad =\gamma\ (\frac{\partial}{\partial x_1}(\xi_{22} + \nabla_2\phi_2))\|\xi + \nabla\phi\|^2 \\
&\qquad\quad + \gamma\ (\xi_{22} + \nabla_2\phi_2)(\frac{\partial}{\partial x_1}\|\xi + \nabla\phi\|^2) \\
&\qquad =\gamma\ \nabla_{21}\phi_2\ \|\xi + \nabla\phi\|^2 \\
&\qquad\quad + \gamma\ (\xi_{22} + \nabla_2\phi_2)(\frac{\partial}{\partial x_1}\|\xi + \nabla\phi\|^2)
\end{aligned} \tag{5.30}$$

Combining Eq.(5.27), Eq.(5.28), Eq.(5.29), and Eq.(5.30), we conclude that

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\frac{\partial}{\partial \xi_{11}} f_\gamma(\xi + \nabla\phi) \\
&=4\ \nabla_{11}\phi_1\|\xi + \nabla\phi\|^2 + 4(\xi_{11} + \nabla_1\phi_1)\frac{\partial}{\partial x_1}\|\xi + \nabla\phi\|^2 \\
&\quad - 2\ \gamma\ \nabla_{11}\phi_1\ det(\xi + \nabla\phi) - 2\ \gamma\ (\xi_{11} + \nabla_1\phi_1))(\frac{\partial}{\partial x_1} det(\xi + \nabla\phi)) \\
&\quad - \gamma\ \nabla_{21}\phi_2\ \|\xi + \nabla\phi\|^2 - \gamma\ (\xi_{22} + \nabla_2\phi_2)(\frac{\partial}{\partial x_1}\|\xi + \nabla\phi\|^2)
\end{aligned} \tag{5.31}$$

We present the results of the other three gradients as well:

$$\begin{aligned}
1)\ &\frac{\partial}{\partial x_2}\frac{\partial}{\partial \xi_{12}} f_\gamma(\xi + \nabla\phi) \\
&= 4\ \nabla_{22}\phi_1\|\xi + \nabla\phi\|^2 + 4(\xi_{12} + \nabla_2\phi_1)\frac{\partial}{\partial x_2}\|\xi + \nabla\phi\|^2 \\
&- 2\ \gamma\ \nabla_{22}\phi_1 det(\xi + \nabla\phi) - 2\ \gamma\ (\xi_{12} + \nabla_2\phi_1))\frac{\partial}{\partial x_2} det(\xi + \nabla\phi) \\
&- \gamma\ \nabla_{12}\phi_2\|\xi + \nabla\phi\|^2 - \gamma\ (\xi_{21} + \nabla_1\phi_1)\frac{\partial}{\partial x_2}\|\xi + \nabla\phi\|^2
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
2)\ &\frac{\partial}{\partial x_1}\frac{\partial}{\partial \xi_{21}} f_\gamma(\xi + \nabla\phi) \\
&= 4\ \nabla_{11}\phi_2\|\xi + \nabla\phi\|^2 + 4(\xi_{21} + \nabla_1\phi_2)\frac{\partial}{\partial x_1}\|\xi + \nabla\phi\|^2 \\
&- 2\ \gamma\ \nabla_{11}\phi_2 det(\xi + \nabla\phi) - 2\ \gamma\ (\xi_{21} + \nabla_1\phi_2))\frac{\partial}{\partial x_1} det(\xi + \nabla\phi) \\
&- \gamma\ \nabla_{21}\phi_1\|\xi + \nabla\phi\|^2 - \gamma\ (\xi_{12} + \nabla_2\phi_1)\frac{\partial}{\partial x_1}\|\xi + \nabla\phi\|^2
\end{aligned} \tag{5.33}$$

$$
\begin{aligned}
3)\ &\frac{\partial}{\partial x_2}\frac{\partial}{\partial \xi_{22}} f_\gamma(\xi+\nabla\phi) \\
&= 4\ \nabla_{22}\phi_2\|\xi+\nabla\phi\|^2 + 4(\xi_{22}+\nabla_2\phi_2)\frac{\partial}{\partial x_2}\|\xi+\nabla\phi\|^2 \\
&- 2\ \gamma\ \nabla_{22}\phi_2 det(\xi+\nabla\phi) - 2\ \gamma\ (\xi_{22}+\nabla_2\phi_2))\frac{\partial}{\partial x_2}det(\xi+\nabla\phi) \\
&- \gamma\ \nabla_{12}\phi_1\|\xi+\nabla\phi\|^2 - \gamma\ (\xi_{11}+\nabla_1\phi_1)\frac{\partial}{\partial x_2}\|\xi+\nabla\phi\|^2
\end{aligned} \tag{5.34}
$$

For the sake of readability, we keep some partial derivatives unexpanded in the above results. For these common terms that appear in all the gradients above, we can write separate functions to calculate them in Matlab to avoid redundancy. We show their expansions below.

$$
\begin{aligned}
\frac{\partial}{\partial x_1}\|\xi+\nabla\phi\|^2 &= 2((\xi_{11}+\nabla_1\phi_1)\nabla_{11}\phi_1 + (\xi_{12}+\nabla_2\phi_1)\nabla_{21}\phi_1 \\
&\quad + (\xi_{21}+\nabla_1\phi_2)\nabla_{11}\phi_2 + (\xi_{22}+\nabla_2\phi_2)\nabla_{21}\phi_2) \\
\frac{\partial}{\partial x_2}\|\xi+\nabla\phi\|^2 &= 2((\xi_{11}+\nabla_1\phi_1)\nabla_{12}\phi_1 + (\xi_{12}+\nabla_2\phi_1)\nabla_{22}\phi_1 \\
&\quad + (\xi_{21}+\nabla_1\phi_2)\nabla_{12}\phi_2 + (\xi_{22}+\nabla_2\phi_2)\nabla_{22}\phi_2) \\
\frac{\partial}{\partial x_1}det(\xi+\nabla\phi) &= \nabla_{11}\phi_1(\xi_{22}+\nabla_2\phi_2) + \nabla_{21}\phi_2(\xi_{11}+\nabla_1\phi_1) \\
&\quad - \nabla_{11}\phi_2(\xi_{12}+\nabla_2\phi_1) - \nabla_{21}\phi_1(\xi_{21}+\nabla_1\phi_2) \\
\frac{\partial}{\partial x_2}det(\xi+\nabla\phi) &= \nabla_{12}\phi_1(\xi_{22}+\nabla_2\phi_2) + \nabla_{22}\phi_2(\xi_{11}+\nabla_1\phi_1) \\
&\quad - \nabla_{12}\phi_2(\xi_{12}+\nabla_2\phi_1) - \nabla_{22}\phi_1(\xi_{21}+\nabla_1\phi_2)
\end{aligned} \tag{5.35}
$$

5.1.2 Determination of Step Sizes

Our next goal is to find a step size $\tau > 0$ in each iteration to guarantee descent. By the definition of the steepest descent,

$$\tau = \underset{\alpha}{\operatorname{argmin}}\ J_\gamma(\phi^k - \alpha\delta J_\gamma(\phi^k)). \tag{5.36}$$

Let's define a new function $\psi(\alpha)$ such that

$$\psi(\alpha) \coloneqq J_\gamma(\phi^k - \alpha\delta J_\gamma(\phi^k)). \tag{5.37}$$

Then finding τ in Eq.(5.36) is equivalent to finding α such that

$$\frac{d}{d\alpha}\psi(\alpha) = 0 \tag{5.38}$$

Expanding $\psi(\alpha)$, we get

$$\frac{d}{d\alpha}\int_\Omega f_\gamma(\xi + \nabla(\phi^k - \alpha\delta J_\gamma(\phi^k)))d\Omega = 0 \tag{5.39}$$

We continue in this fashion, obtaining

$$\frac{d\psi(\alpha)}{d\alpha} = \int_\Omega \left\langle Df_\gamma(\xi + \nabla(\phi^k - \alpha\delta J_\gamma(\phi^k)), \frac{\partial}{\partial\alpha}(\nabla(\phi^k - \alpha\delta J_\gamma(\phi^k))) \right\rangle d\Omega \tag{5.40}$$

which finally gives us

$$\frac{d\psi(\alpha)}{d\alpha} = \int_\Omega \langle \delta J_\gamma(\phi^k - \alpha\delta J_\gamma(\phi^k)), \delta J_\gamma(\phi^k)\rangle d\Omega. \tag{5.41}$$

We denote $\dfrac{d\psi(\alpha)}{d\alpha}$ by $g(\alpha)$, and the problem becomes finding the critical point of $g(\alpha)$ However, solving for $g(\alpha) = 0$ directly would be complicated and time consuming, so we take advantage of the secant method which is in the form of

$$\alpha^{n+1} = \alpha^n - g(\alpha^n)\frac{\alpha^n - \alpha^{n-1}}{g(\alpha^n) - g(\alpha^{n-1})}, \tag{5.42}$$

with initial guesses α^0 and α^1 selected as two random numbers that are close to 0.

5.1.3 Numerical Implementations

We now turn to our numerical method with a hope in finding a set of appropriate γ, ξ, and funcional ϕ that proves Morrey's Conjecture, i.e., $J_\gamma(\phi) - f_\gamma(\xi) < 0$.

First we introduce our numerical approximations. Let n be a positive number and $h = \dfrac{1}{n}$. We partition Ω into $(n+1)^2$ number of identical squares $\Omega_{ij} = [ih, (i+1)h] \times [jh, (j+1)h]$, $0 \le i, j \le n-1$ as shown in Figure 2 below.
To generate those mesh nodes and record their coordinates, we take advantage of a built-in MATLAB function *meshgrid* which creates a $(n+1)^2 \times 2$ vector containing all the coordinates of the nodes that is in the form

$$\begin{bmatrix} X_0 & Y_0 \\ X_0 & Y_1 \\ \vdots & \vdots \\ X_0 & Y_n \\ X_1 & Y_0 \\ X_1 & Y_1 \\ \vdots & \vdots \\ X_n & Y_{n-1} \\ X_n & Y_n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & h \\ \vdots & \vdots \\ 0 & nh \\ h & 0 \\ h & h \\ \vdots & \vdots \\ nh & (n-1)h \\ nh & nh \end{bmatrix} \tag{5.43}$$

We build this mesh in order to approximate gradients using the Finite Difference method, then we can sum up all the values on each nodes to approximate the surface integral in J_γ. We will first demonstrate how we use MATLAB and the Finite Difference method to obtain derivatives on each node. With the initial assumption that function $\phi(x, y)$ is a continuous function with all necessary derivatives exist in Ω, we write

$$\begin{aligned} \phi_{i+1,j} &= \phi(x+h, y) \\ \phi_{i,j+1} &= \phi(x, y+h) \end{aligned} \tag{5.44}$$

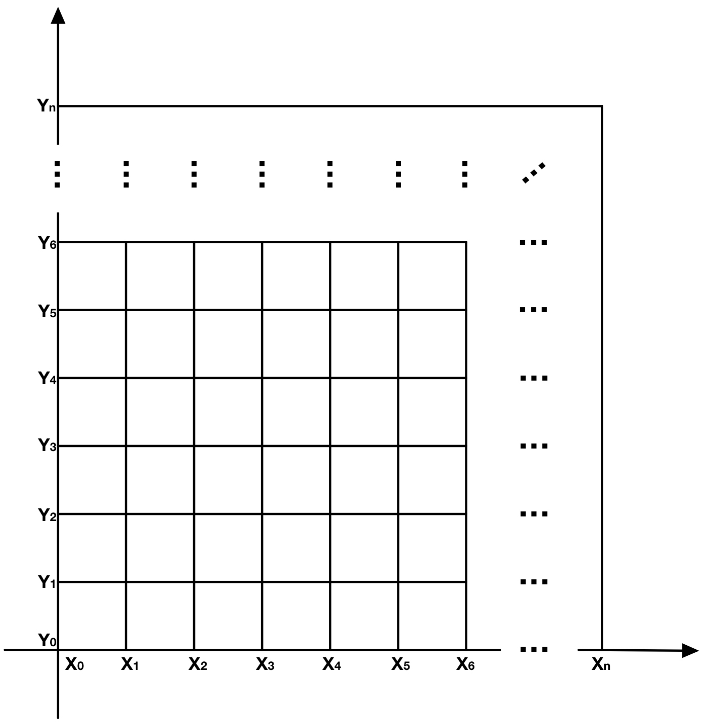

Figure 2: Mesh Grid Example

Thus, by the Forward Finite Difference Approximation, for an arbitrary Lipschitz function f with two variables x and y

$$\begin{aligned}\nabla_1 f(x,y) &= \frac{\partial f(x,y)}{\partial x} = \frac{f(x+h,y)-f(x,y)}{h} \\ \nabla_2 f(x,y) &= \frac{\partial f(x,y)}{\partial y} = \frac{f(x,y+h)-f(x,y)}{h}\end{aligned} \tag{5.45}$$

It is critical to test the precision of our approximation method. Choose a random function, we can manually solve for and calculate its partial derivatives. These real derivative values are then compared to our approximations. Using the criterion of maximum errors and mean square errors, we demonstrate the accuracy of our method. We choose ϕ as

$$\phi(x,y) = \begin{bmatrix}\sin(\pi(x+2y)) & \cos(\pi(3x+y))\end{bmatrix} \tag{5.46}$$

The construction of the Finite Difference method itself tells us that the smaller the h is, the more accurate the approximations should be. We observe that, for $h = 10^{-4}$, our approximated partial derivatives are pretty close to the real values.

	Max Error	MSE
$\nabla_1\phi_1$	0.00049348	$1.2176 \cdot 10^{-7}$
$\nabla_1\phi_2$	0.0044413	$9.8627 \cdot 10^{-6}$
$\nabla_2\phi_1$	0.0019739	$1.9482 \cdot 10^{-6}$
$\nabla_2\phi_2$	0.00049348	$1.2176 \cdot 10^{-7}$

As for the second derivatives, we can still utilize the Finite Difference method. However, there are two cases.

1. For $\nabla_{11} f(x,y)$ and $\nabla_{22} f(x,y)$, we apply the second-order central method.

$$\begin{aligned}\nabla_{11} f(x,y) &= \frac{\partial^2 f(x,y)}{\partial x \partial x} = \frac{f(x+h,y)-2f(x,y)+f(x-h,y)}{h^2} \\ \nabla_{22} f(x,y) &= \frac{\partial^2 f(x,y)}{\partial y \partial y} = \frac{f(x,y+h)-2f(x,y)+f(x,y-h)}{h^2}\end{aligned} \tag{5.47}$$

2. For $\nabla_{12} f(x,y)$ and $\nabla_{21} f(x,y)$, it is obvious that, by the Clairaut's theorem,

$$\nabla_{12} f(x,y) = \nabla_{21} f(x,y) \tag{5.48}$$

Then based on what we already have for the first order derivatives, we apply the forward method again.

$$\nabla_{12} f(x,y) = \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\nabla_1 f(x,y+h) - \nabla_1 f(x,y)}{h} \tag{5.49}$$

Up to this point, we have everything needed to calculate the updating factor for our steepest descent, which is

$$\delta J_\gamma(\phi^k) = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} \frac{\partial}{\partial \xi_{ij}} f_\gamma(\xi + \nabla\phi^k). \tag{5.50}$$

5.2 Numerical Results

In this part, we present our numerical results. In what follows, we introduce a new notation for the expression we are interested in:

$$d_\gamma(\phi^k) = \int_\Omega \left[f_\gamma(\xi + \nabla\phi) - f_\gamma(\xi)\right] dx, \tag{5.51}$$

and because Ω was defined to be $(0,1) \times (0,1)$, the equation Eq.(5.51) becomes

$$d_\gamma(\phi^k) = J_\gamma(\phi^k) - f_\gamma(\xi). \tag{5.52}$$

We separate our results into two cases with respect the value of γ: it stays constant or it changes value after each iteration. Also, to avoid mesh effects that make our integral extraordinarily large, we decide to use a mesh size of $h = 0.1$ throughout this part.

5.2.1 Simulations with γ fixed

In the first part, we choose

$$\phi^0(x,y) = \left[\sin(x(x-1)y(y-1)), \sin(x(x-1)y(y-1))^2\right] \tag{5.53}$$

with a randomly generated 2×2 matrix as our ξ.

$$\xi = \begin{bmatrix} 0.682296 & 0.920074 \\ 0.335120 & 0.736268 \end{bmatrix} \tag{5.54}$$

We start with $\gamma = \dfrac{4}{\sqrt{3}}$ and run 200 iterations without changing it. We then decrease the value of γ and run another 200 iterations. We repeat this process until γ reaches 2. The outcomes are shown below:

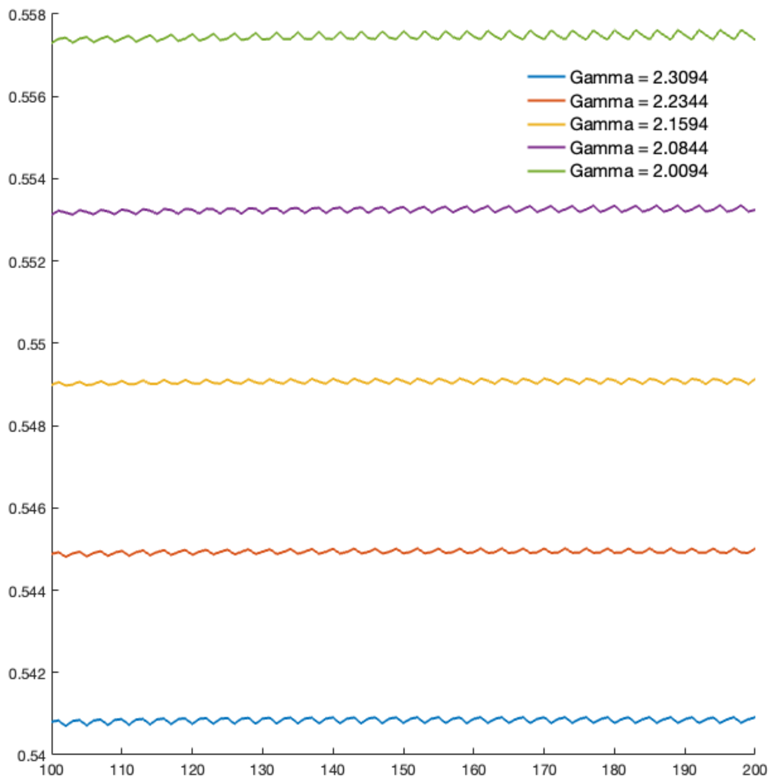

Figure 3: $d_\gamma(\phi^k)$ with different γ

5.2.2 Simulations with Various γ

Then we consider the case where γ changes. Similarly, we start with $\gamma = \dfrac{4}{\sqrt{3}}$ and reduce its value until 2. However, this time it is reduced after each iteration.

5.2.2.1 Results with $\xi = [1, 0; 0, \sqrt{3}]$

1. For

$$\phi_1^0(x, y) = \left[\sin(x(x-1)y(y-1)), \sin(x(x-1)y(y-1))^2\right],$$

we get

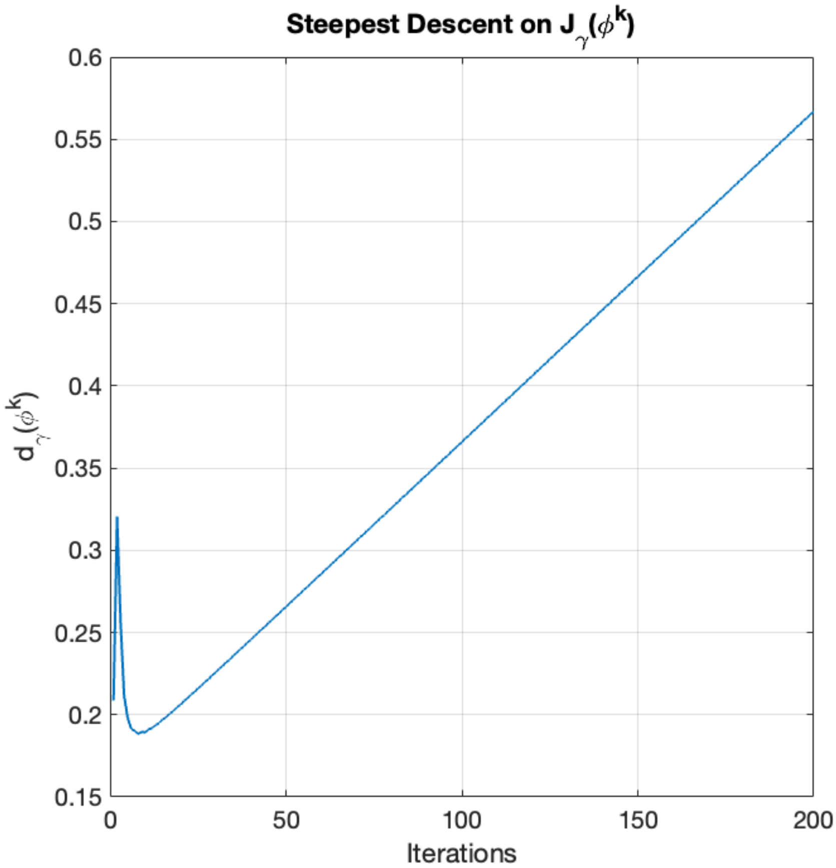

Figure 4: Steepest descent on ϕ_1 with fixed $\xi = [1, 0; 0, \sqrt{3}]$ and changing γ

2. For

$$\phi_2^0(x,y) = \left[(x(x-1)y(y-1)), 0\right],$$

we get

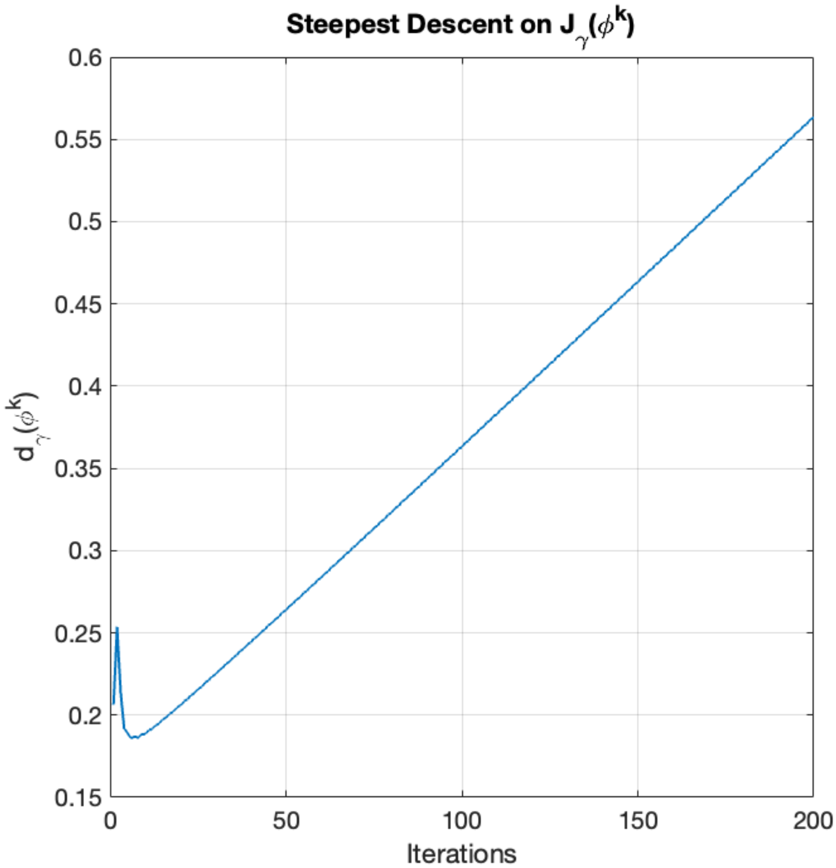

Figure 5: Steepest descent on ϕ_2 with fixed $\xi = [1, 0; 0, \sqrt{3}]$ and changing γ

3. For
$$\phi_3^0(x,y) = \left[\frac{1}{2\pi}\sin(2\pi x), \frac{1}{2\pi}\sin(2\pi y)\right],$$
we get

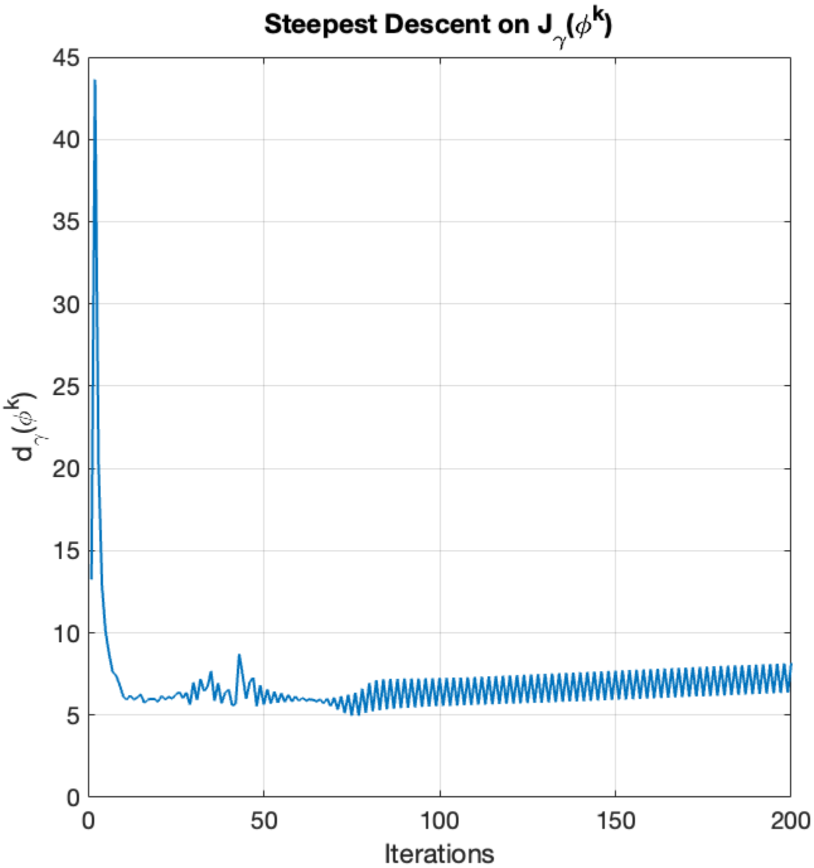

Figure 6: Steepest descent on ϕ_3 with fixed $\xi = [1, 0; 0, \sqrt{3}]$ and changing γ

4. For

$$\phi_4^0(x,y) = \left[\frac{1}{100}\sin(\pi x), \frac{1}{100}\sin(\frac{3\pi}{2}y)\right],$$

we get

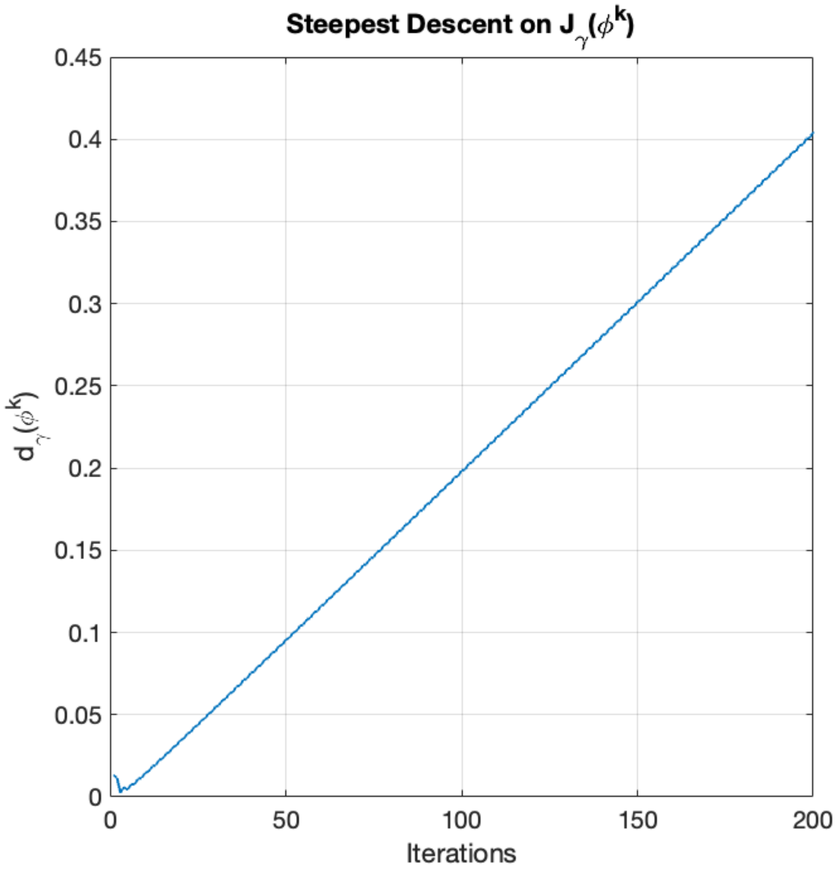

Figure 7: Steepest descent on ϕ_4 with fixed $\xi = [1, 0; 0, \sqrt{3}]$ and changing γ

5.2.2.2 Results with $\xi = [0, 0; 0, 0]$

1. For

$$\phi_1^0(x, y) = \left[\sin(x(x-1)y(y-1)), \sin(x(x-1)y(y-1))^2\right],$$

we get

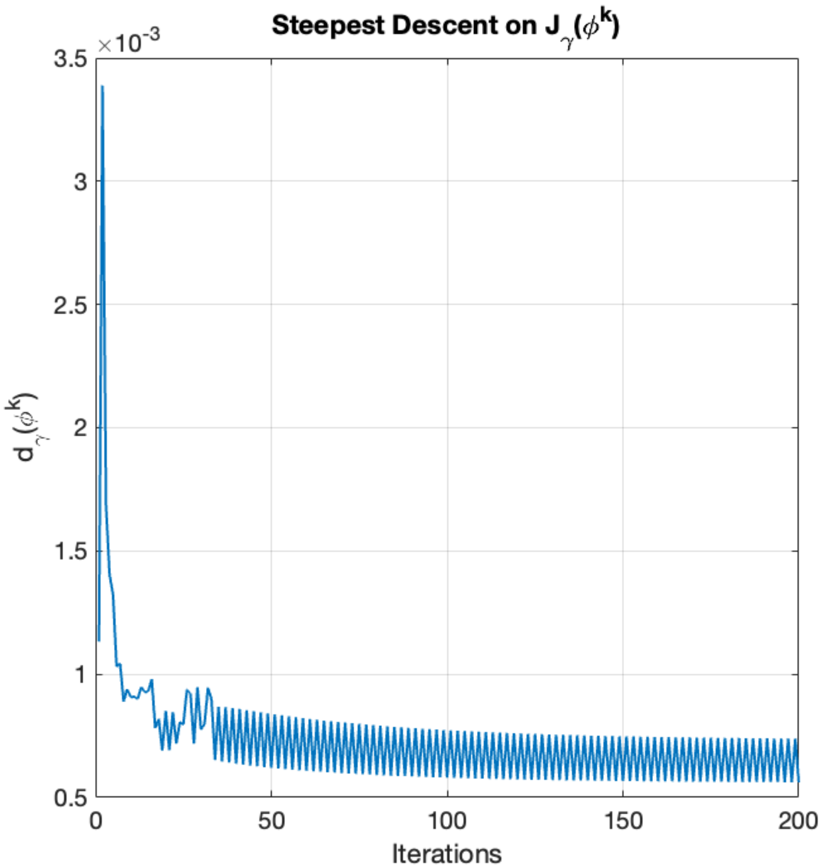

Figure 8: Steepest descent on ϕ_1 with fixed $\xi = [0, 0; 0, 0]$ and changing γ

2. For

$$\phi_2^0(x,y) = \left[(x(x-1)y(y-1)),0\right],$$

we get

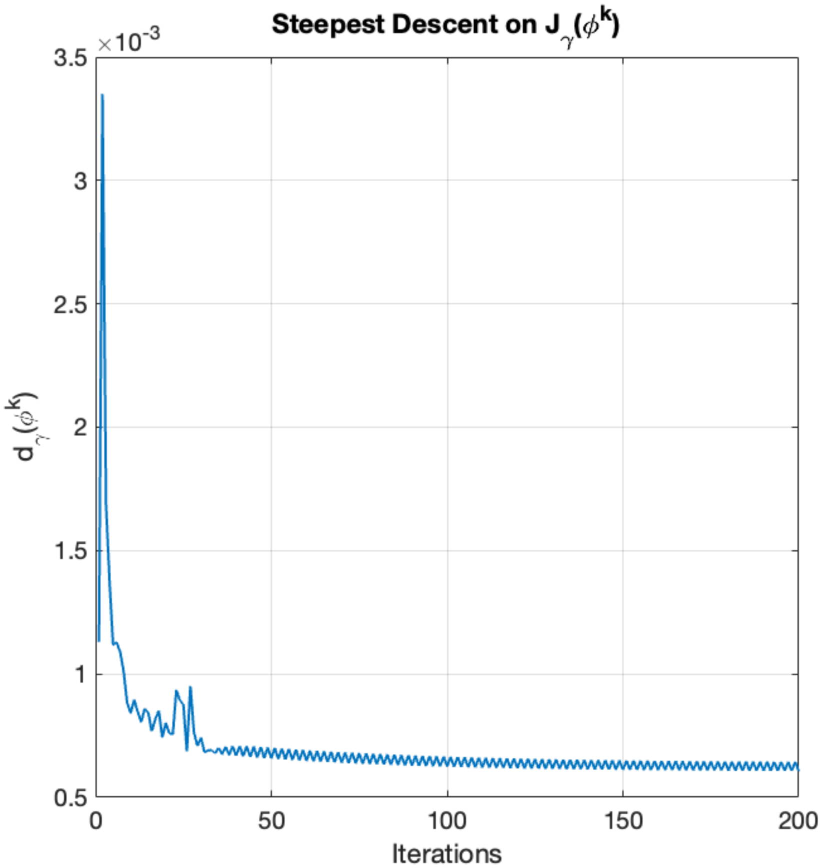

Figure 9: Steepest descent on ϕ_2 with fixed $\xi = [0,0;0,0]$ and changing γ

3. For

$$\phi_3^0(x,y) = \left[\frac{1}{2\pi}\sin(2\pi x), \frac{1}{2\pi}\sin(2\pi y)\right],$$

we get

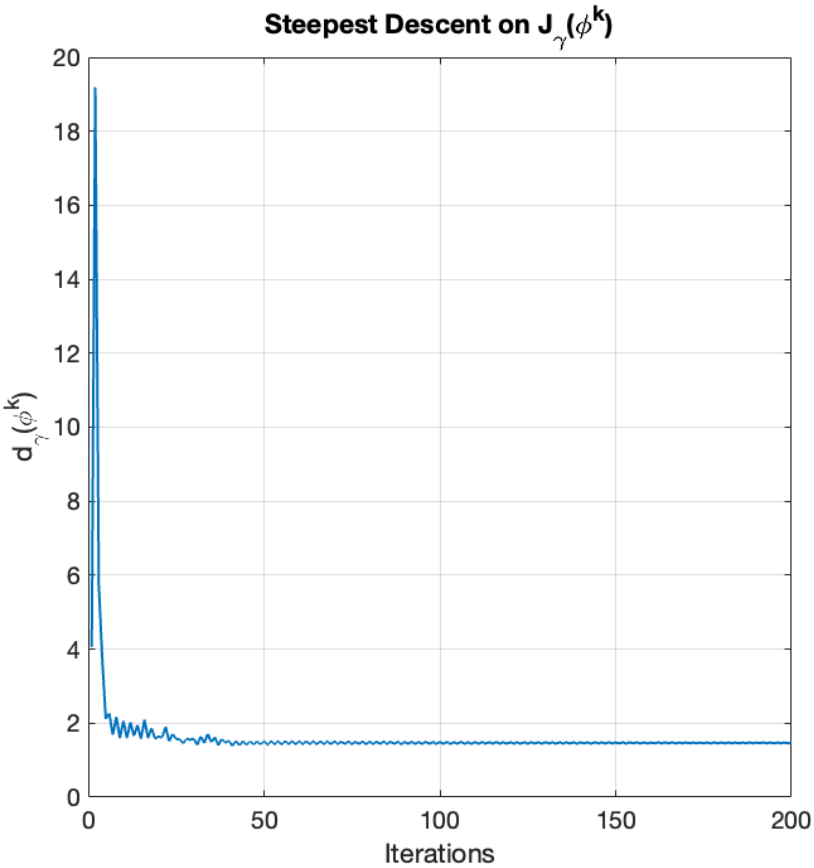

Figure 10: Steepest descent on ϕ_3 with fixed $\xi = [0, 0; 0, 0]$ and changing γ

4. For

$$\phi_4^0(x,y) = \left[\frac{1}{100}\sin(\pi x), \frac{1}{100}\sin(\frac{3\pi}{2}y)\right],$$

we get

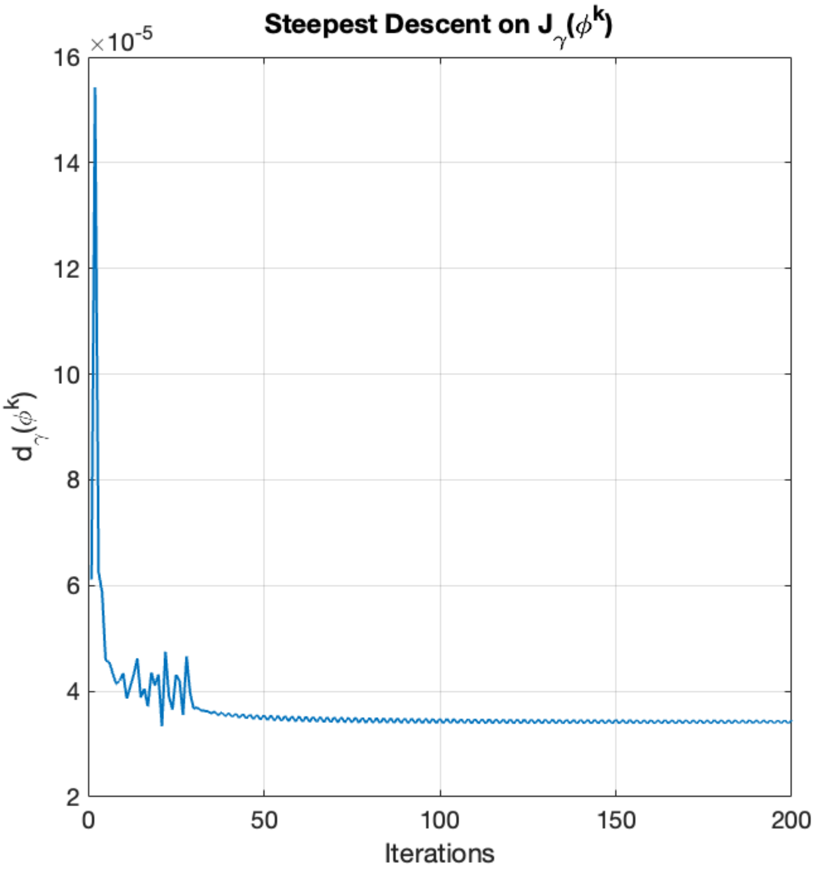

Figure 11: Steepest descent on ϕ_4 with fixed $\xi = [0,0;0,0]$ and changing γ

5.2.2.3 Results with $\xi = [0, -1; 1, 0]$

1. For

$$\phi_1^0(x,y) = \left[\sin(x(x-1)y(y-1)), \sin(x(x-1)y(y-1))^2\right],$$

we get

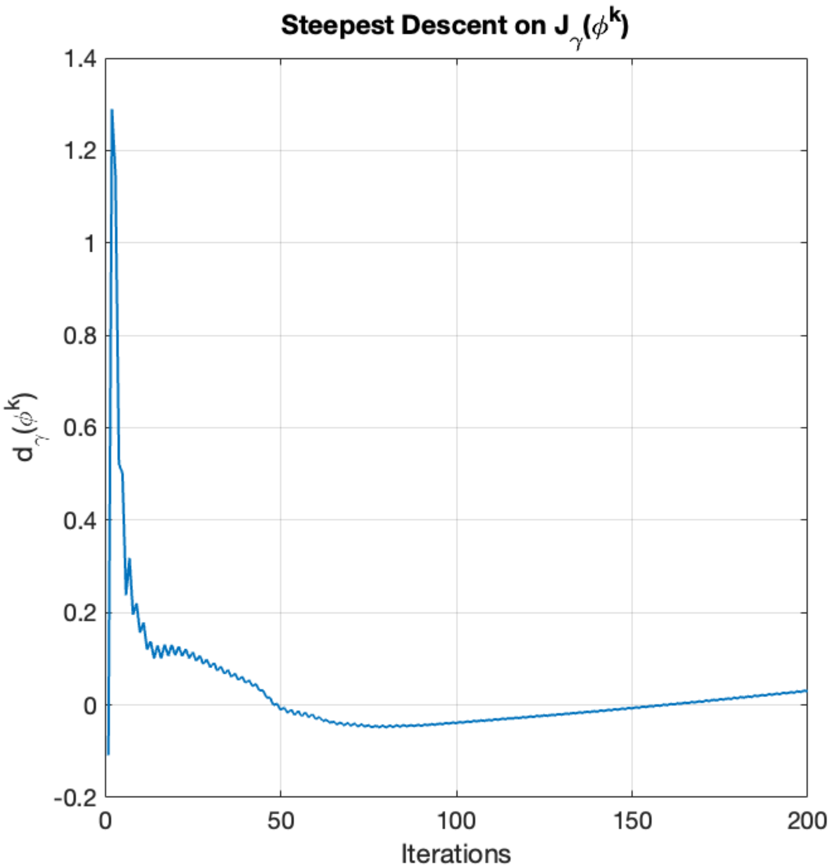

Figure 12: Steepest descent on ϕ_1 with fixed $\xi = [0, -1; 1, 0]$ and changing γ

2. For
$$\phi_2^0(x,y) = \left[(x(x-1)y(y-1)), 0\right],$$
we get

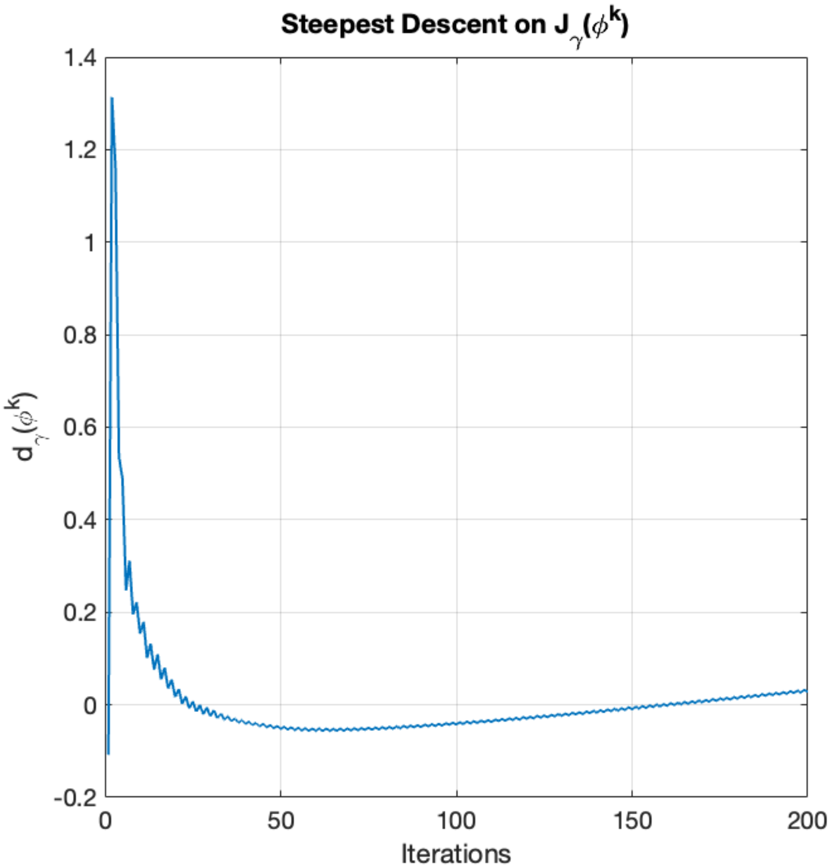

Figure 13: Steepest descent on ϕ_2 with fixed $\xi = [0, -1; 1, 0]$ and changing γ

3. For

$$\phi_3^0(x,y) = \left[\frac{1}{2\pi}\sin(2\pi x), \frac{1}{2\pi}\sin(2\pi y)\right],$$

we get

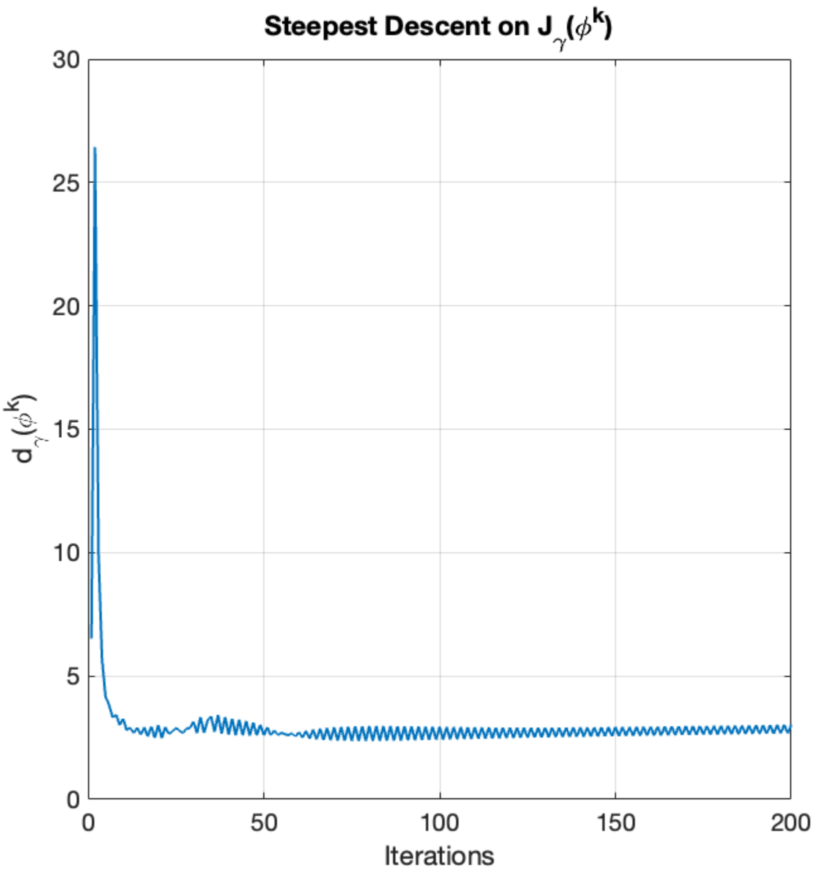

Figure 14: Steepest descent on ϕ_3 with fixed $\xi = [0, -1; 1, 0]$ and changing γ

4. For

$$\phi_4^0(x,y) = \left[\frac{1}{100}\sin(\pi x), \frac{1}{100}\sin(\frac{3\pi}{2}y)\right],$$

we get

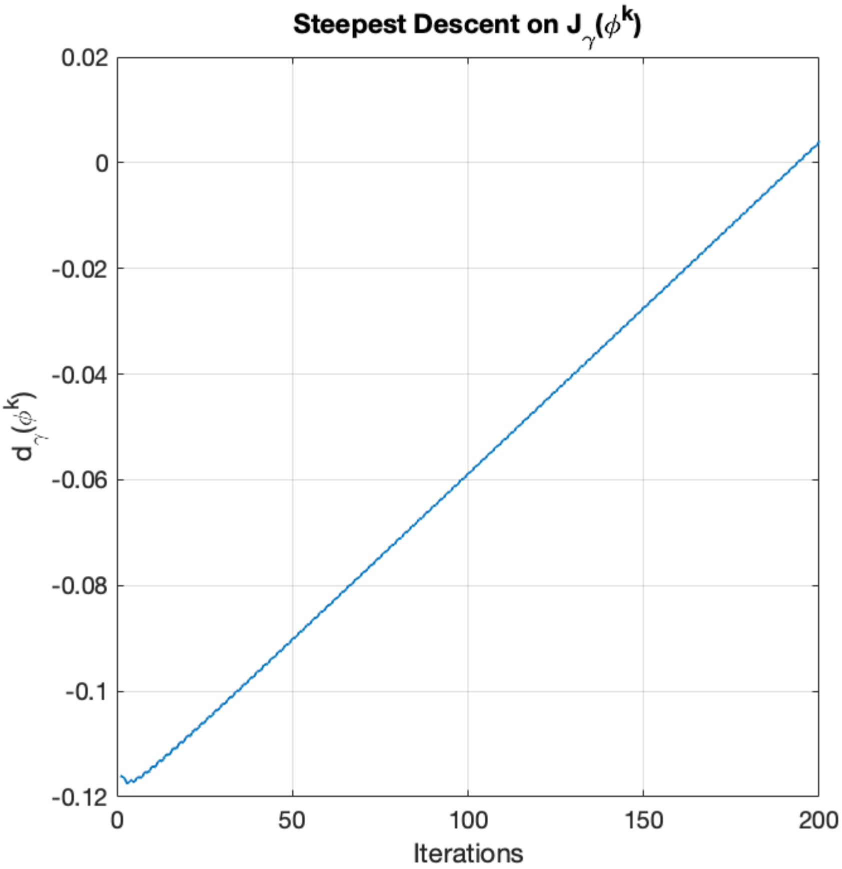

Figure 15: Steepest descent on ϕ_4 with fixed $\xi = [0, -1; 1, 0]$ and changing γ

5.2.2.4 Results with $\xi = [1, 0; 0, \sqrt{2.8}]$

1. For

$$\phi_1^0(x, y) = \left[\sin(x(x-1)y(y-1)), \sin(x(x-1)y(y-1))^2\right],$$

we get

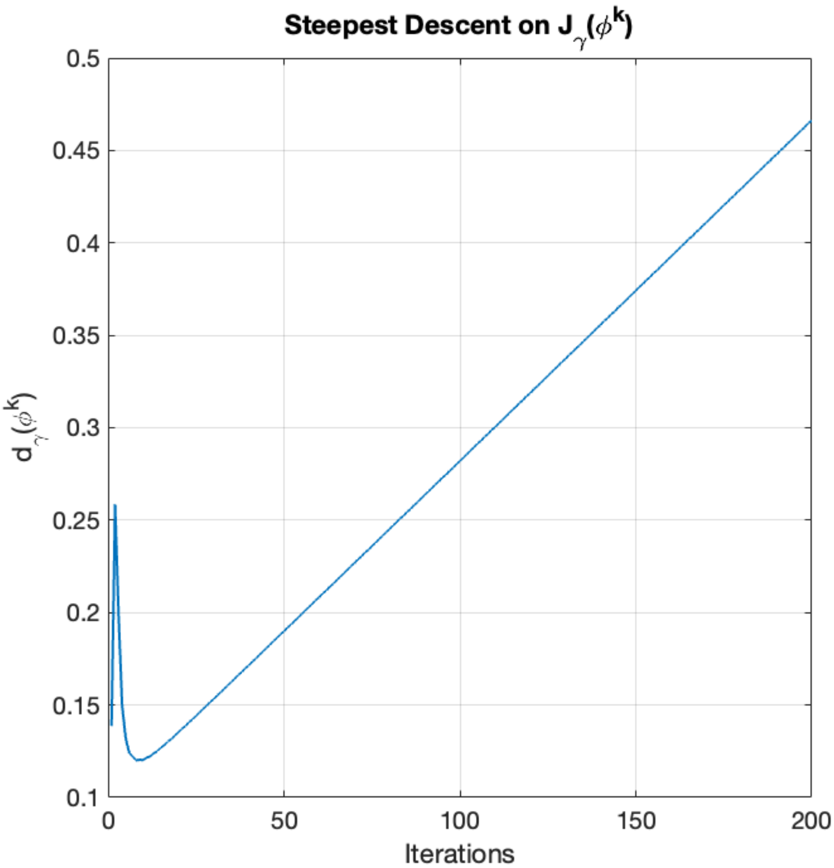

Figure 16: Steepest descent on ϕ_1 with fixed $\xi = [1, 0; 0, \sqrt{2.8}]$ and changing γ

2. For

$$\phi_2^0(x,y) = \left[(x(x-1)y(y-1)), 0\right],$$

we get

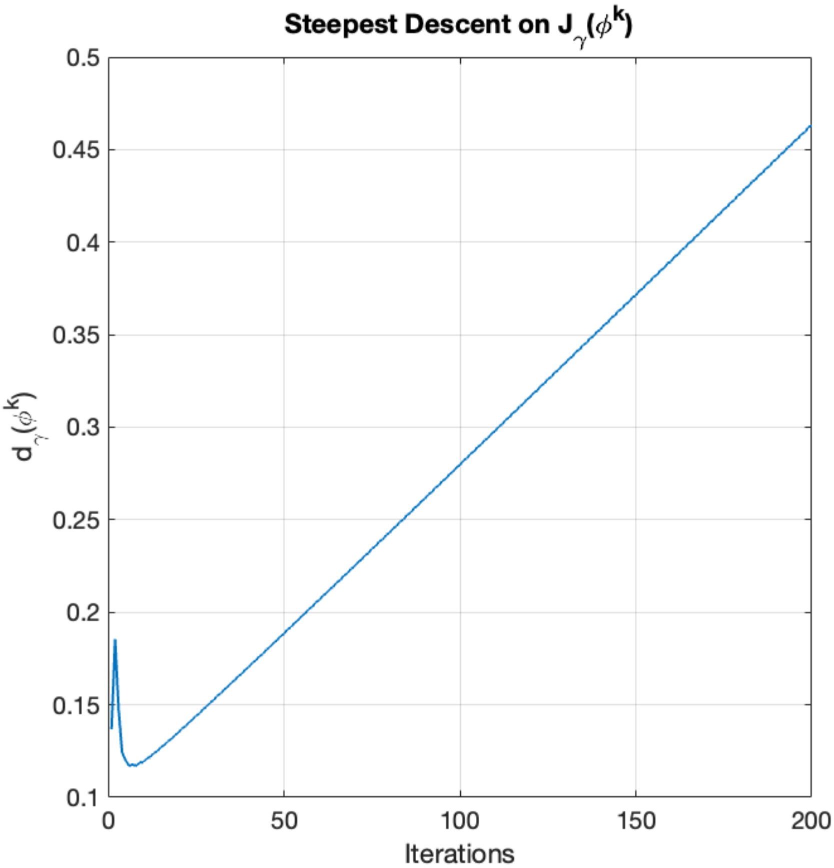

Figure 17: Steepest descent on ϕ_2 with fixed $\xi = [1, 0; 0, \sqrt{2.8}]$ and changing γ

3. For
$$\phi^0(x,y) = \left[\frac{1}{2\pi}\sin(2\pi x), \frac{1}{2\pi}\sin(2\pi y)\right],$$
we get

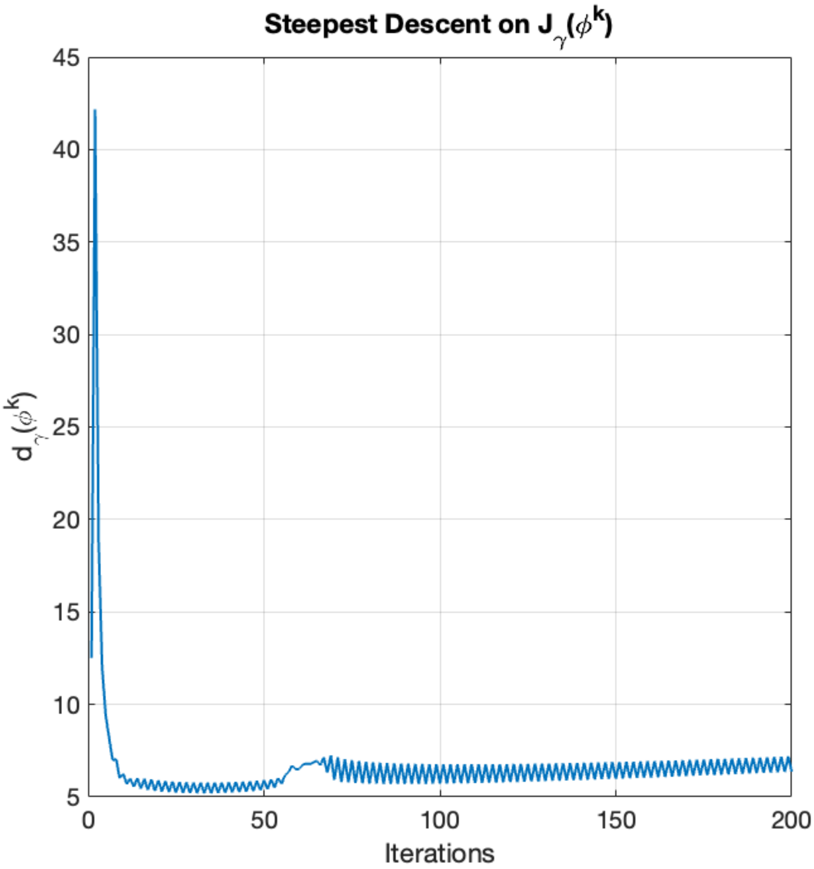

Figure 18: Steepest descent on ϕ_3 with fixed $\xi = [1,0;0,\sqrt{2.8}]$ and changing γ

4. For

$$\phi^0(x,y) = \Big[\frac{1}{100}\sin(\pi x), \frac{1}{100}\sin(\frac{3\pi}{2}y)\Big],$$

we get

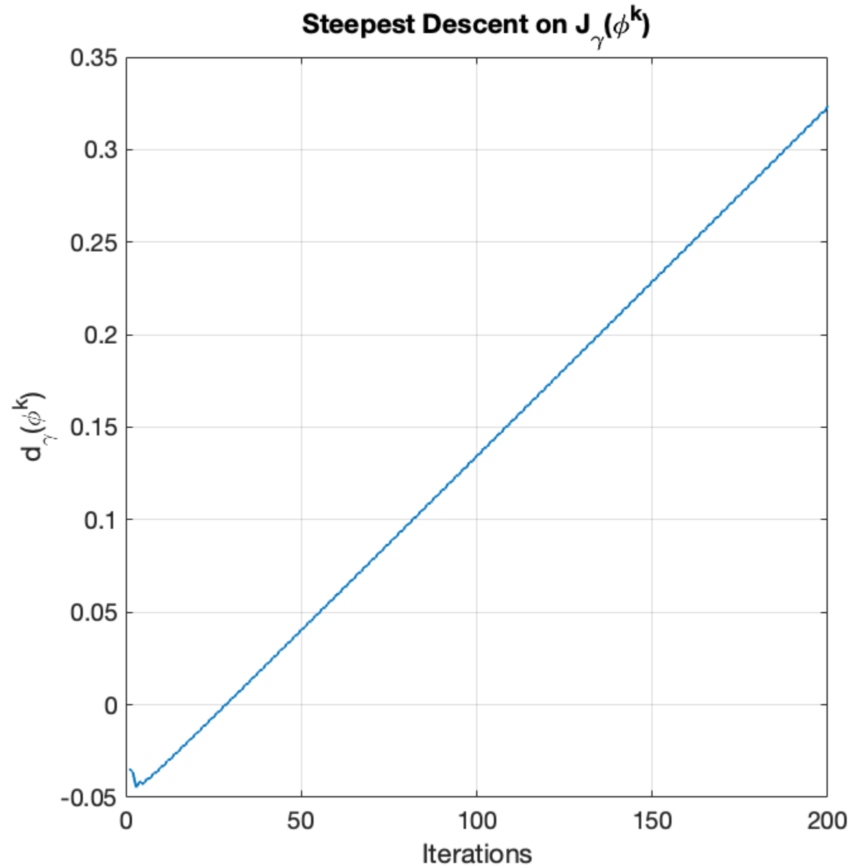

Figure 19: Steepest descent on ϕ_4 with fixed $\xi = [1, 0; 0, \sqrt{2.8}]$ and changing γ

5.2.2.5 Results with ξ randomly generated at the beginning of the steepest descent iterations

ξ is a 2×2 matrix of uniformly distributed random numbers between 0 and 1.

1. For

$$\phi_1^0(x,y) = \left[\sin(x(x-1)y(y-1)), \sin(x(x-1)y(y-1))^2\right]$$

and

$$\xi = \begin{bmatrix} 0.678735 & 0.743132 \\ 0.757740 & 0.392227 \end{bmatrix}$$

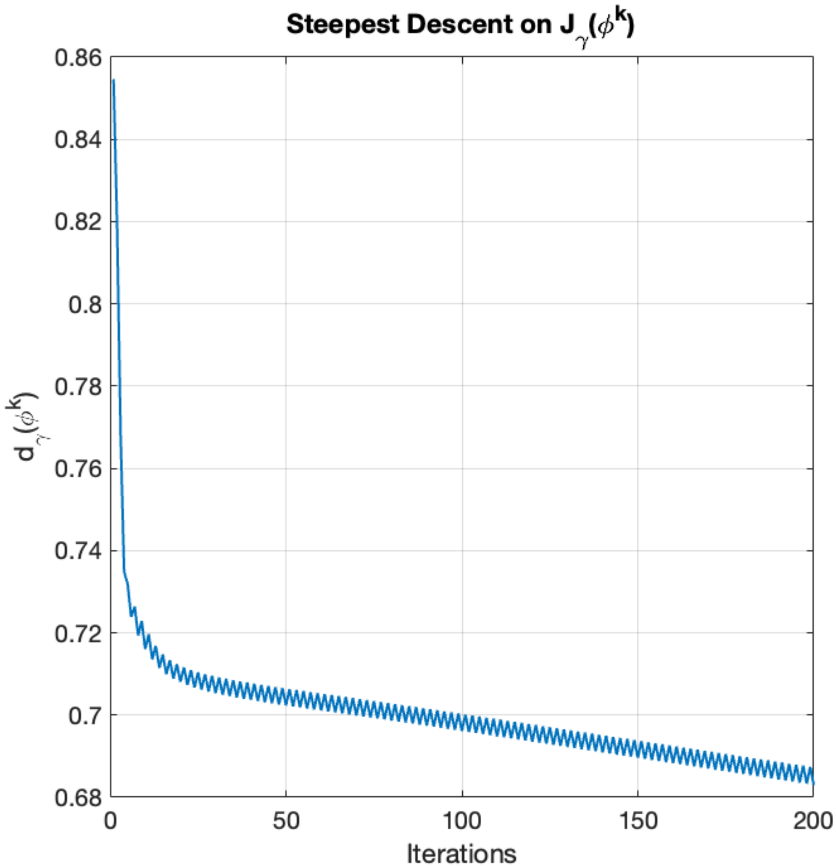

Figure 20: Steepest descent on ϕ_1 with fixed randomly generated ξ and changing γ

2. For

$$\phi_2^0(x,y) = \big[(x(x-1)y(y-1)), 0\big]$$

and

$$\xi = \begin{bmatrix} 0.655478 & 0.706046 \\ 0.171187 & 0.031833 \end{bmatrix}$$

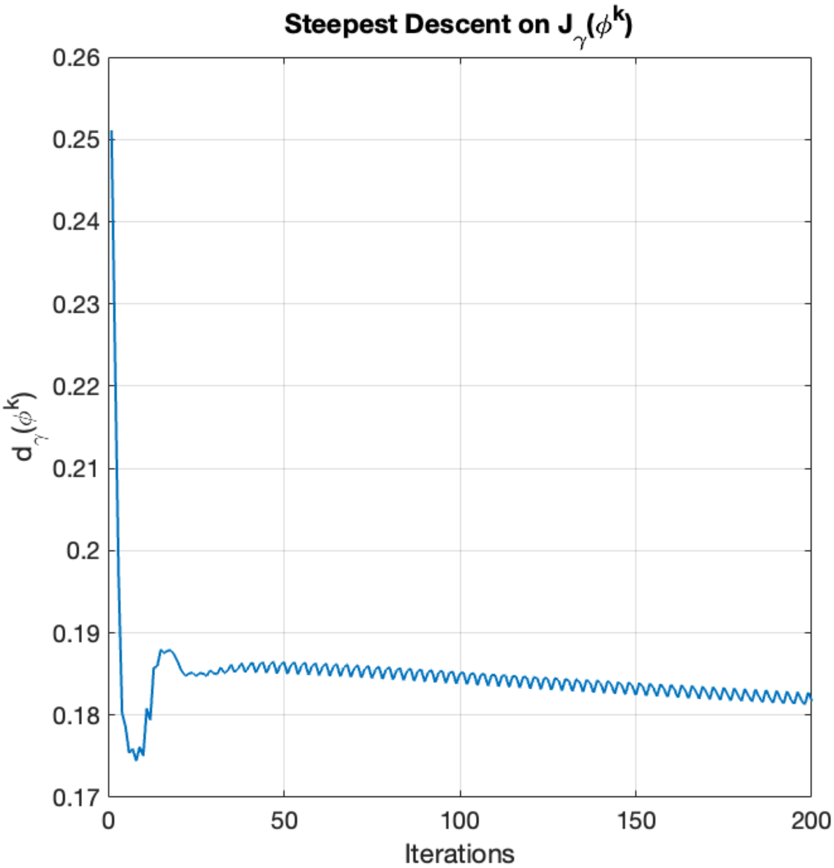

Figure 21: Steepest Descent on ϕ_2 with fixed randomly generated ξ and changing γ

3. For

$$\phi_3^0(x,y) = \left[\frac{1}{2\pi}\sin(2\pi x), \frac{1}{2\pi}\sin(2\pi y)\right]$$

and

$$\xi = \begin{bmatrix} 0.694828 & 0.950222 \\ 0.317099 & 0.034446 \end{bmatrix}$$

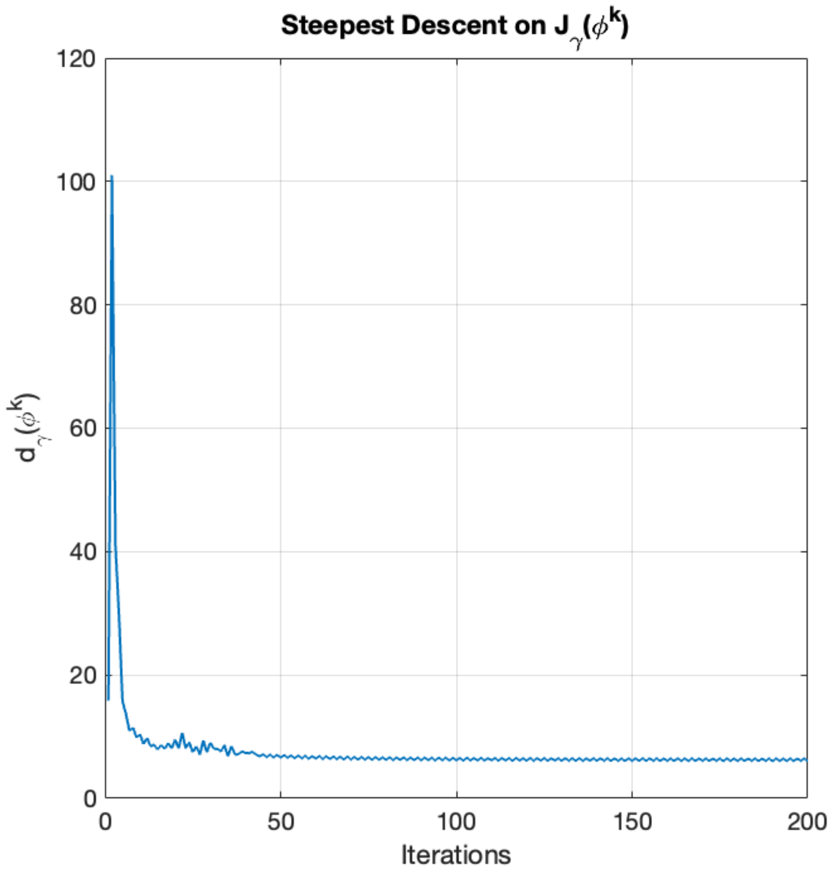

Figure 22: Steepest descent on ϕ_3 with fixed randomly generated ξ and changing γ

4. For

$$\phi_4^0(x,y) = \Big[\frac{1}{100}\sin(\pi x), \frac{1}{100}\sin(\frac{3\pi}{2}y)\Big]$$

and

$$\xi = \begin{bmatrix} 0.709365 & 0.276025 \\ 0.754686 & 0.679703 \end{bmatrix}$$

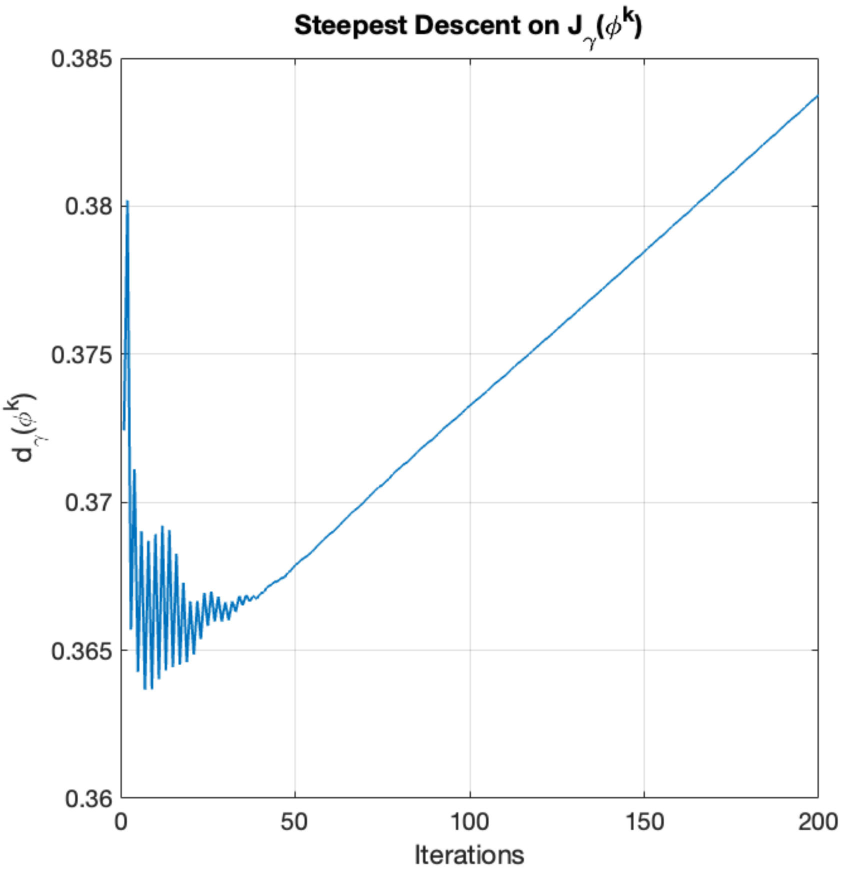

Figure 23: Steepest descent on ϕ_4 with fixed randomly generated ξ and changing γ

5.2.2.6 Results with ξ randomly generated at the beginning of each iteration

1. For

$$\phi_1^0(x,y) = \left[\sin(x(x-1)y(y-1)), \sin(x(x-1)y(y-1))^2\right],$$

we get

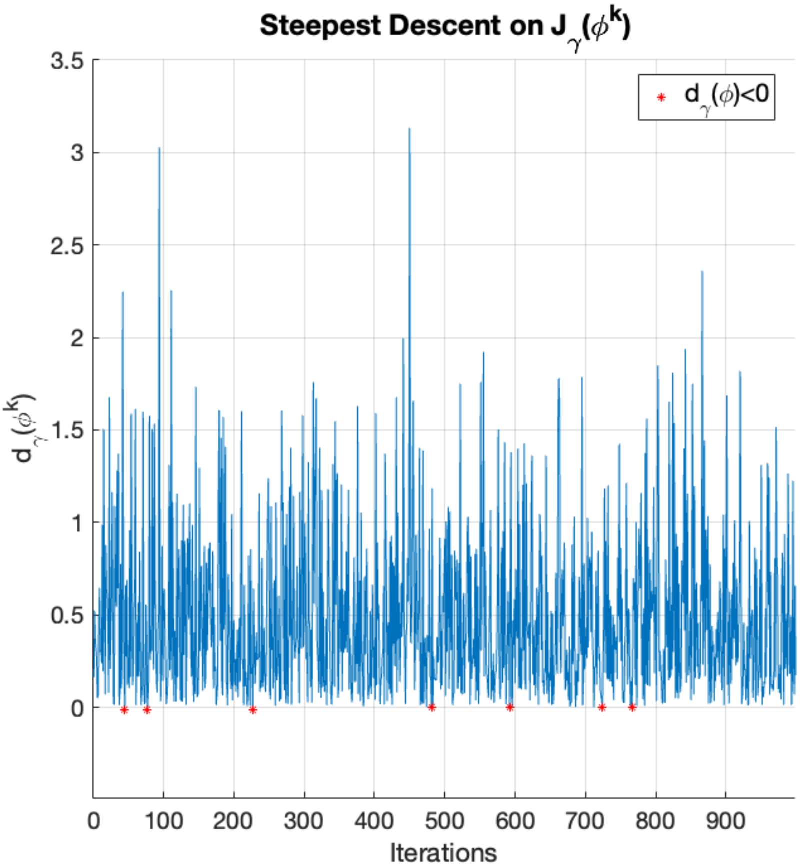

Figure 24: Steepest descent on ϕ_1 with ξ randomly generated in each iteration and changing γ

Table 1: Numerical values of $\xi, \gamma, J_\gamma(\phi_k)$ and $d_\gamma(\phi_k)$ when $d_\gamma(\phi_k) < 0$

	ξ	γ	$J_\gamma(\phi_k)$	$d_\gamma(\phi_k)$
1.	$\begin{bmatrix} 0.66684 & 0.03839 \\ 0.097573 & 0.65968 \end{bmatrix}$	2.2958	-0.11131	-0.012894
2.	$\begin{bmatrix} 0.66771 & 0.12654 \\ 0.049398 & 0.9548 \end{bmatrix}$	2.2859	-0.10471	-0.01239
3.	$\begin{bmatrix} 0.72935 & 0.086371 \\ 0.10256 & 0.88553 \end{bmatrix}$	2.2395	-0.13993	-0.016581
4.	$\begin{bmatrix} 0.75449 & 0.043381 \\ 0.0062739 & 0.58269 \end{bmatrix}$	2.1609	-0.036257	-0.0010055
5.	$\begin{bmatrix} 0.78112 & 0.039509 \\ 0.0056236 & 0.616 \end{bmatrix}$	2.1265	-0.031675	-0.00039584
6.	$\begin{bmatrix} 0.60279 & 0.079766 \\ 0.026003 & 0.64121 \end{bmatrix}$	2.0857	-0.015927	-7.4392e-05
7.	$\begin{bmatrix} 0.75449 & 0.043381 \\ 0.0062739 & 0.58269 \end{bmatrix}$	2.1609	-0.036257	-0.0010055

The table above lists the values of ξ and γ when the Jensen inequality is violated, i.e. $d_\gamma(\phi) < 0$, during the iterations.

2. For

$$\phi_2^0(x,y) = \left[(x(x-1)y(y-1)), 0\right],$$

we get

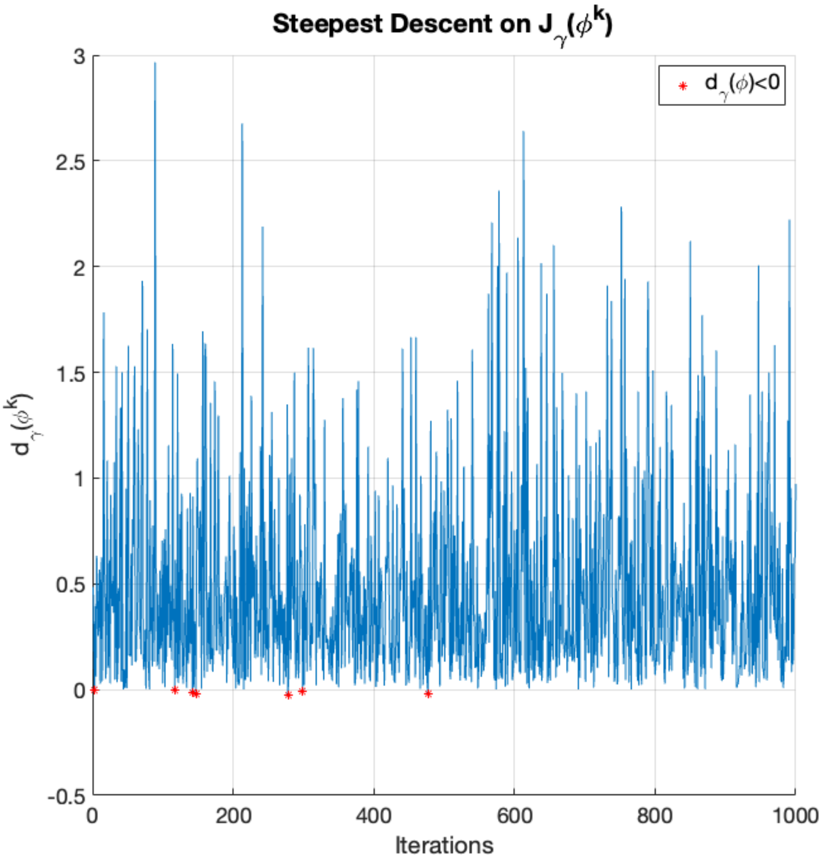

Figure 25: Steepest descent on ϕ_2 with ξ randomly generated in each iteration changing γ

Table 2: Numerical values of $\xi, \gamma, J_\gamma(\phi_k)$ and $d_\gamma(\phi_k)$ when $d_\gamma(\phi_k) < 0$

	ξ	γ	$J_\gamma(\phi_k)$	$d_\gamma(\phi_k)$
1.	$\begin{bmatrix} 0.98189 & 0.33618 \\ 0.088792 & 0.92305 \end{bmatrix}$	2.3088	-0.17268	-0.0049726
2.	$\begin{bmatrix} 0.95611 & 0.32192 \\ 0.018587 & 0.834 \end{bmatrix}$	2.2735	-0.15057	-0.0038753
3.	$\begin{bmatrix} 0.76279 & 0.060225 \\ 0.19213 & 0.86044 \end{bmatrix}$	2.2658	-0.14815	-0.014408
4.	$\begin{bmatrix} 0.8279 & 0.027321 \\ 0.26833 & 0.96424 \end{bmatrix}$	2.2642	-0.19228	-0.018429
5.	$\begin{bmatrix} 0.81574 & 0.011821 \\ 0.1179 & 0.84962 \end{bmatrix}$	2.2237	-0.21947	-0.027838
6.	$\begin{bmatrix} 0.89173 & 0.1669 \\ 0.026521 & 0.72494 \end{bmatrix}$	2.2172	-0.10633	-0.0061888
7.	$\begin{bmatrix} 0.90207 & 0.032499 \\ 0.1886 & 0.99237 \end{bmatrix}$	2.1621	-0.17816	-0.018311

3. For

$$\phi_3^0(x,y) = \left[\frac{1}{2\pi}\sin(2\pi x), \frac{1}{2\pi}\sin(2\pi y)\right],$$

we get

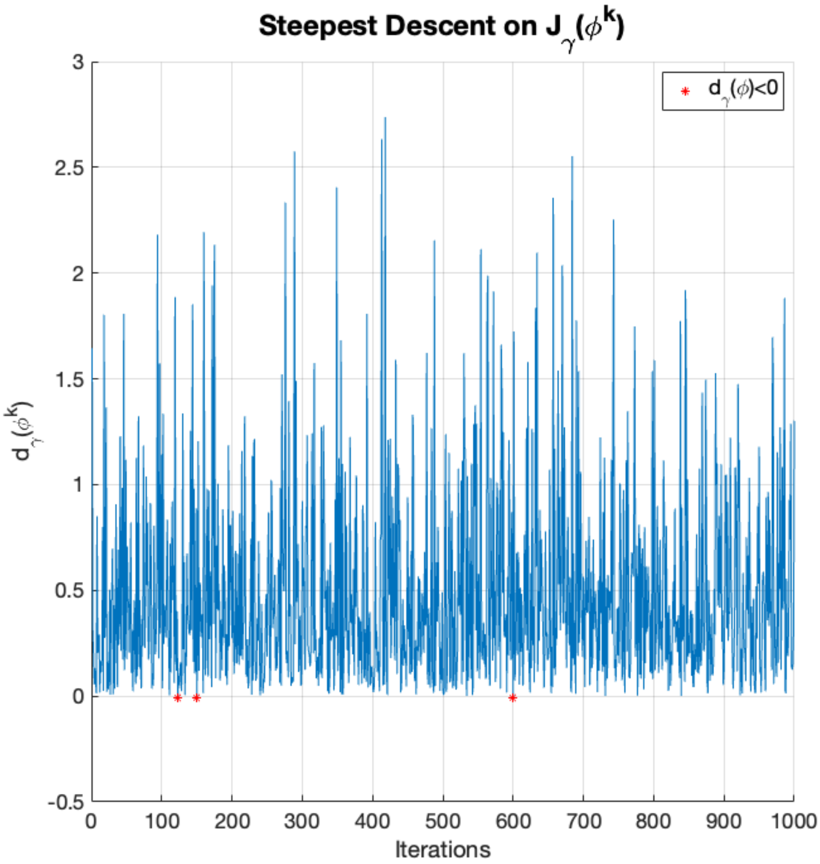

Figure 26: Steepest descent on ϕ_3 with ξ randomly generated in each iteration changing γ

Table 3: Numerical values of $\xi, \gamma, J_\gamma(\phi_k)$ and $d_\gamma(\phi_k)$ when $d_\gamma(\phi_k) < 0$

	ξ	γ	$J_\gamma(\phi_k)$	$d_\gamma(\phi_k)$
1.	$\begin{bmatrix} 0.66064 & 0.074988 \\ 0.0058409 & 0.57225 \end{bmatrix}$	2.2717	-0.073426	-0.005525
2.	$\begin{bmatrix} 0.70309 & 0.038669 \\ 0.19216 & 0.70708 \end{bmatrix}$	2.2633	-0.084793	-0.0066755
3.	$\begin{bmatrix} 0.96291 & 0.072563 \\ 0.10086 & 0.86599 \end{bmatrix}$	2.1241	-0.11515	-0.0083836

4. For
$$\phi_4^0(x,y) = \left[\frac{1}{100}\sin(\pi x), \frac{1}{100}\sin(\frac{3\pi}{2}y)\right],$$
we get

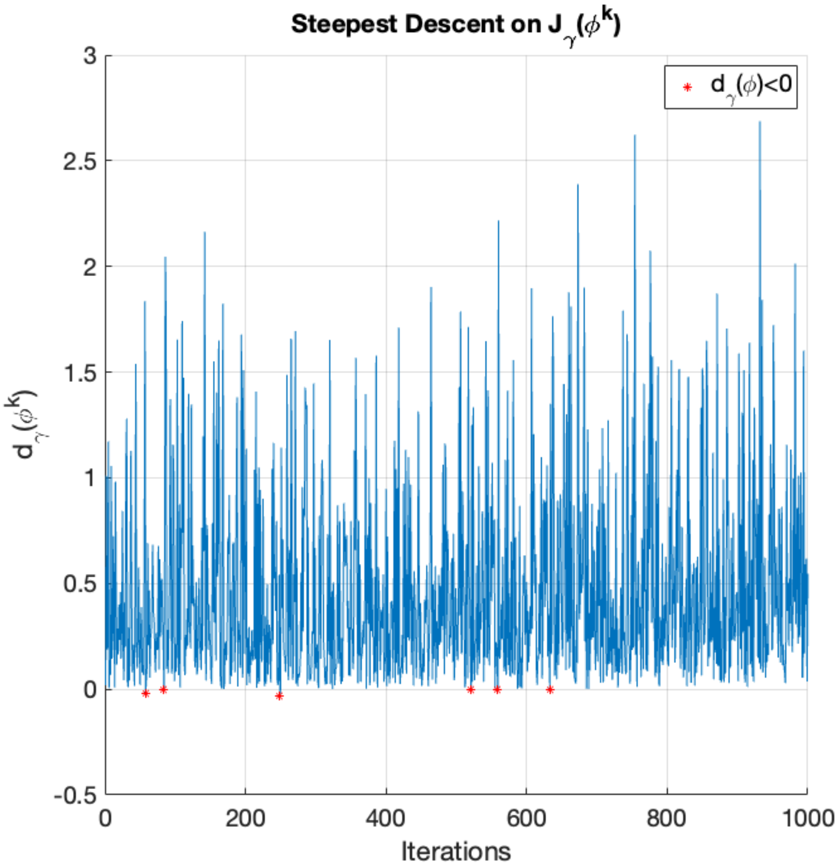

Figure 27: Steepest descent on ϕ_4 with ξ randomly generated in each iteration changing γ

Table 4: Numerical values of $\xi, \gamma, J_\gamma(\phi_k)$ and $d_\gamma(\phi_k)$ when $d_\gamma(\phi_k) < 0$

	ξ	γ	$J_\gamma(\phi_k)$	$d_\gamma(\phi_k)$
1.	$\begin{bmatrix} 0.7246 & 0.01548 \\ 0.20019 & 0.82154 \end{bmatrix}$	2.2918	-0.16219	-0.017237
2.	$\begin{bmatrix} 0.66072 & 0.28593 \\ 0.0055909 & 0.8434 \end{bmatrix}$	2.2837	-0.049032	-0.00071716
3.	$\begin{bmatrix} 0.81874 & 0.021442 \\ 0.11408 & 0.8742 \end{bmatrix}$	2.2327	-0.24057	-0.031294
4.	$\begin{bmatrix} 0.82189 & 0.038704 \\ 0.038139 & 0.68668 \end{bmatrix}$	2.1485	-0.07333	-0.0050091
5.	$\begin{bmatrix} 0.45435 & 0.0023352 \\ 0.015027 & 0.40725 \end{bmatrix}$	2.1368	-0.0086804	-0.00019427
6.	$\begin{bmatrix} 0.6931 & 0.0099955 \\ 0.014925 & 0.92606 \end{bmatrix}$	2.1139	-0.026195	-0.0018741

5.3 Gradient Descent Algorithm

In this part, we only consider the case when $\xi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Inspired by the Duke University's research [10], we can derive a new constraint for γ when $f_\gamma(\xi)$ is quasi-convex.

The definition of quasi-convexity is given by

$$\int_\Omega f_\gamma(\xi + \nabla\phi) - f_\gamma(\xi) d\Omega > 0 \tag{5.55}$$

We continue to use the Dacorogna and Marcellini's example function [12], so we get

$$\int_\Omega \|\xi + \nabla\phi\|^4 - \|\xi + \nabla\phi\|^2 \gamma det(\xi + \nabla\phi) - \|\xi\|^4 + \|\xi\|^2 \gamma det(\xi) d\Omega > 0 \tag{5.56}$$

We plug $\xi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ into the above ineqaulity,

$$\int_\Omega \|\nabla\phi\|^4 - \|\nabla\phi\|^2 \gamma det(\nabla\phi) d\Omega > 0 \tag{5.57}$$

$$\gamma > \frac{\int_\Omega \|\nabla\phi\|^4}{\int_\Omega \|\nabla\phi\|^2 det(\nabla\phi)} \tag{5.58}$$

Thus,

$$\gamma = \sup_{\phi \in W_0^{1,\infty}(\Omega,\mathbb{R})} \left\{ \frac{\int_\Omega \|\nabla\phi\|^4}{\int_\Omega \|\nabla\phi\|^2 det(\nabla\phi)} \right\} \tag{5.59}$$

Finding the top bound for γ is then equivalent to finding the minimal value for function h which we define below

$$h = -\frac{\int_\Omega \|\nabla\phi\|^4}{\int_\Omega \|\nabla\phi\|^2 det(\nabla\phi)}. \tag{5.60}$$

5.3.1 Expressions of function ϕ and plots of its surfaces

For the mappings ϕ, we consider separable functions in the form of $g(x)\cos(y)$, $g(x)\sin(y)$ or simply $g(x)u(y)$ where

$$g(x) = a_0 + \sum_{i=1}^{n} a_i \sin(i\ x) + \sum_{j=1}^{n} b_j \cos(j\ x) \tag{5.61}$$

and

$$u(y) = c_0 + \sum_{i=1}^{n} c_i \sin(i\ y) + \sum_{j=1}^{n} d_j \cos(j\ y). \tag{5.62}$$

In the following figures, we plot the surfaces of $g(x)\cos(y)$ and $g(x)\sin(y)$ which may entail some useful information.

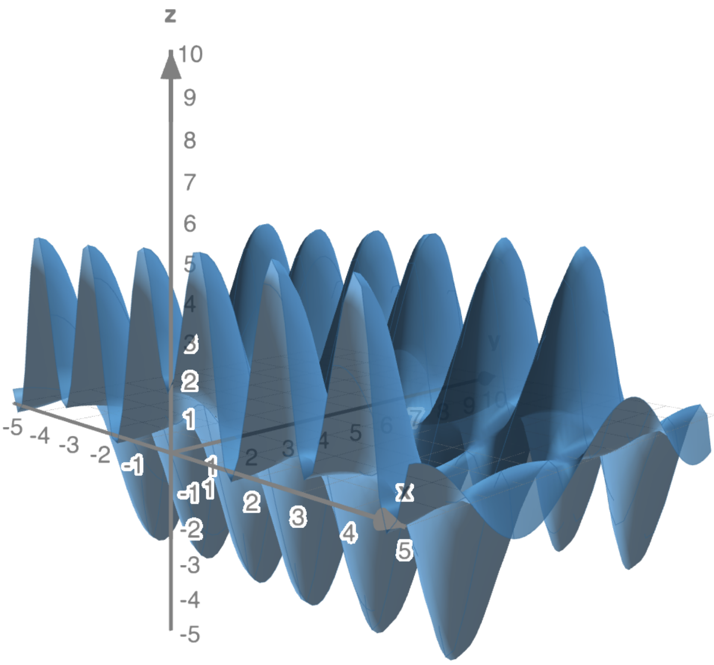

Figure 28: Illustration of the surface $g(x)\cos(y)$

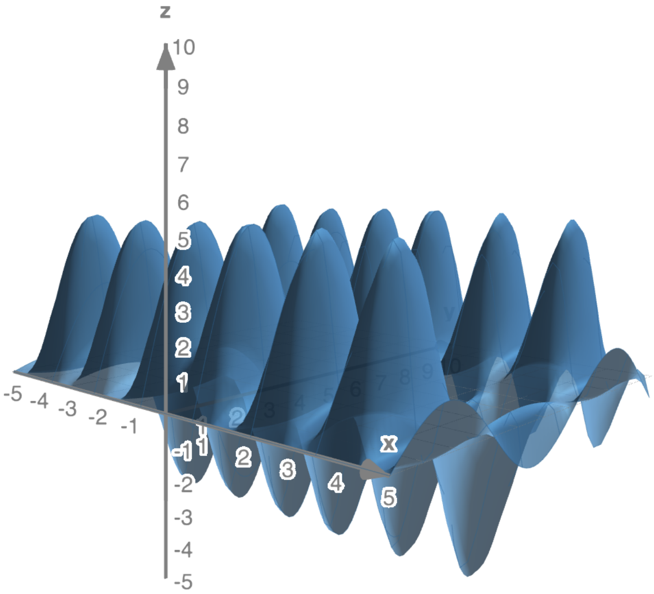

Figure 29: Illustration of the surface $g(x)\sin(y)$

5.4 Numerical Results

We first consider

$$\phi_1(x, y) = \begin{bmatrix} g(x)\cos(y) \\ \sin(y) \end{bmatrix}. \tag{5.63}$$

We can turn $h(x)$ in Eq.(5.60) into a function of the variables $x = [a_0, a_1, \dots, a_n, b_1, \dots, b_n]$. The gradient descent algorithm is then

$$x^{k+1} = x^k - \tau \nabla f(x^k) \tag{5.64}$$

First we let $n = 4$ in $g(x)$, the results show that the minimum value of γ we can find is around 5.0052.

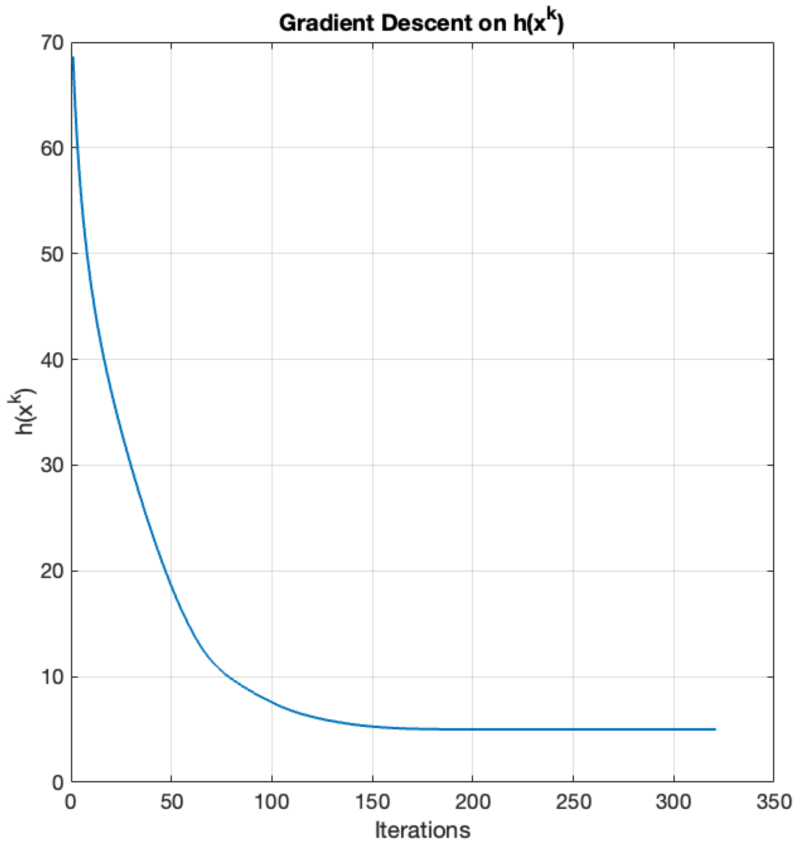

Figure 30: Gradient descent on $h(x)$ with $n = 4$

The optimal value for γ is around 5.0052 which means that with the ϕ we check, $f_\gamma(\phi)$ is quasi-convex if $\gamma < 5.0052$.

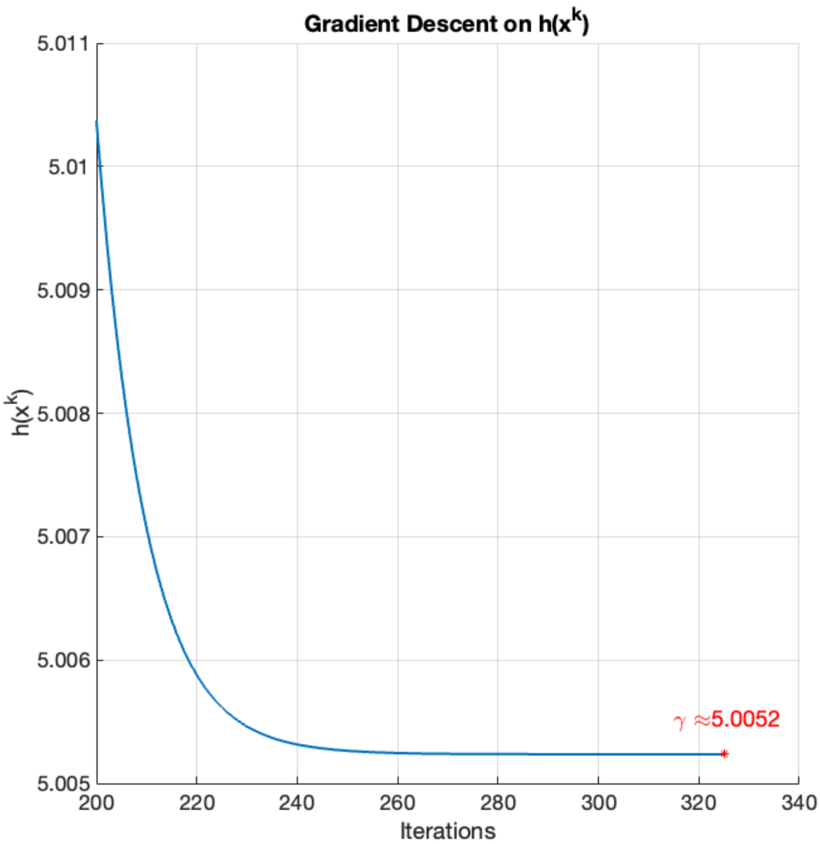

Figure 31: Gradient descent on $h(x)$ with $n = 4$ for last larger iterations

Then we increase the number of coefficients to 11, i.e. $n = 5$.

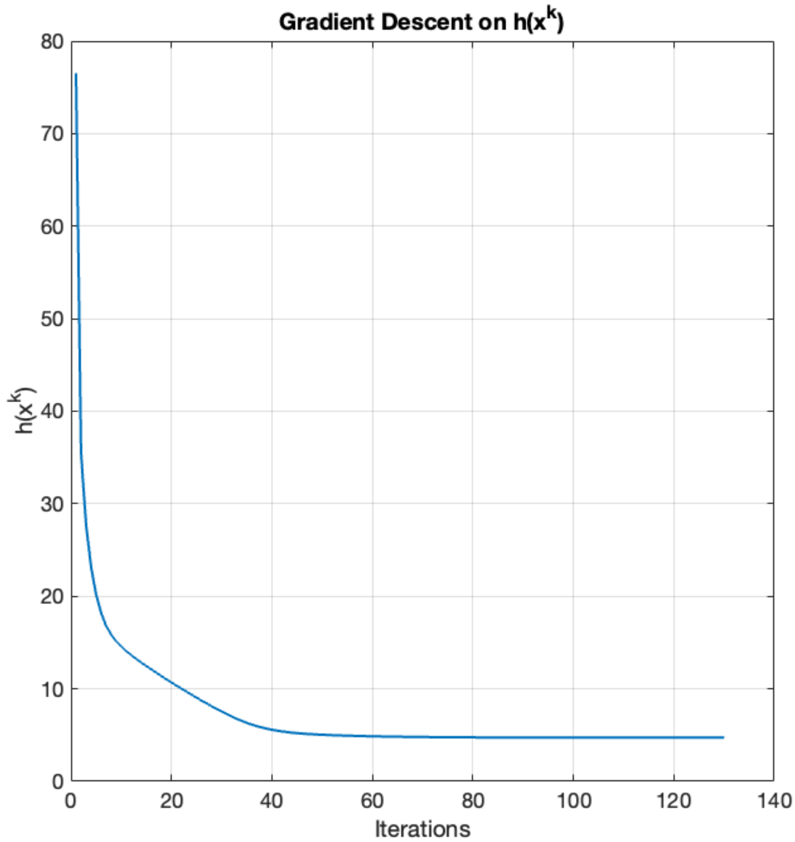

Figure 32: Gradient descent on $h(x)$ with $n = 5$

We can see that the descending speed gets slower and γ finally converges at about 4.7225.

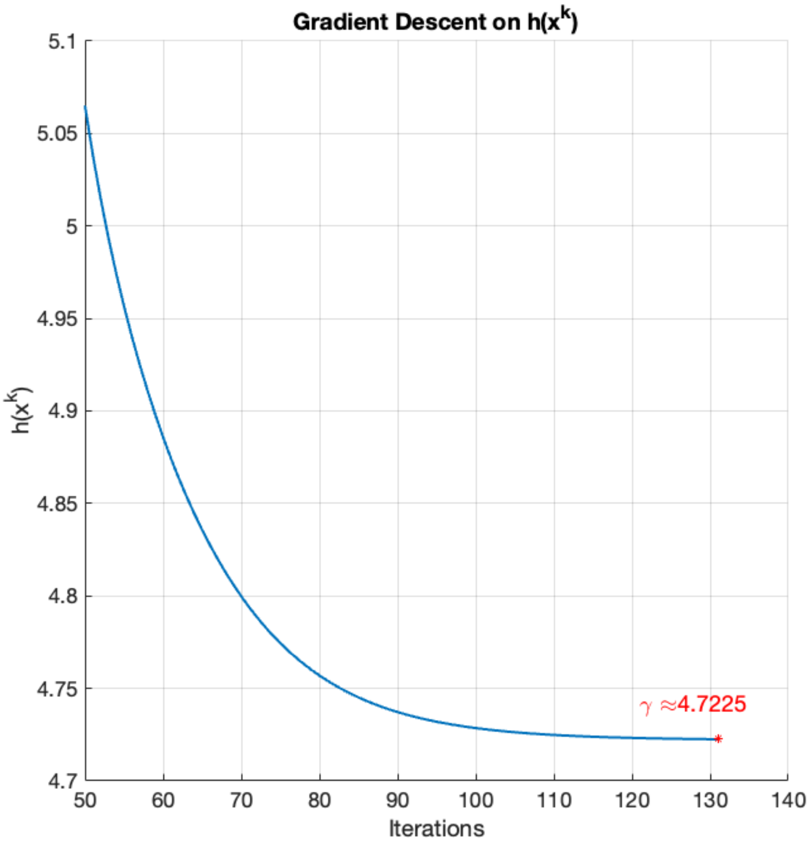

Figure 33: Gradient descent on $h(x)$ with $n = 5$ for last larger iterations

However, since 4.7225 is far from $\dfrac{4}{\sqrt{3}}$, we decide to continue increasing the number of coefficients in the function $g(x)$ in Eq.(5.63) to $n = 6$.

$$g(x) = a_0 + \sum_{i=1}^{6} a_i \sin(i\ x) + \sum_{j=1}^{6} b_j \cos(j\ x) \tag{5.65}$$

The results are shown below. With 13 coefficients, γ converges at 4.5958.

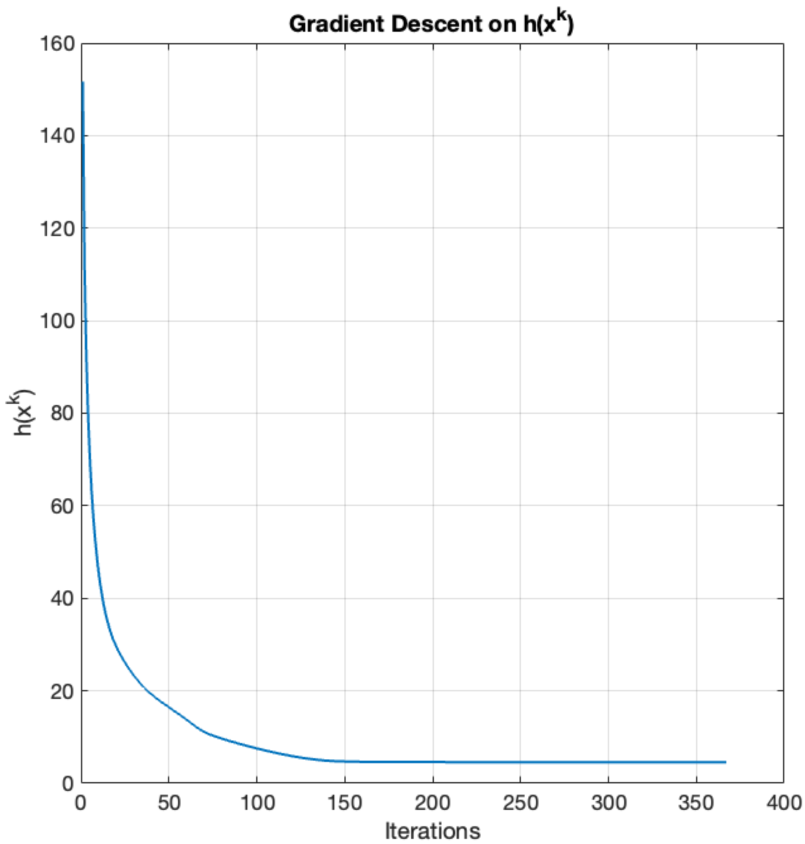

Figure 34: Gradient descent on (x) with $n = 6$

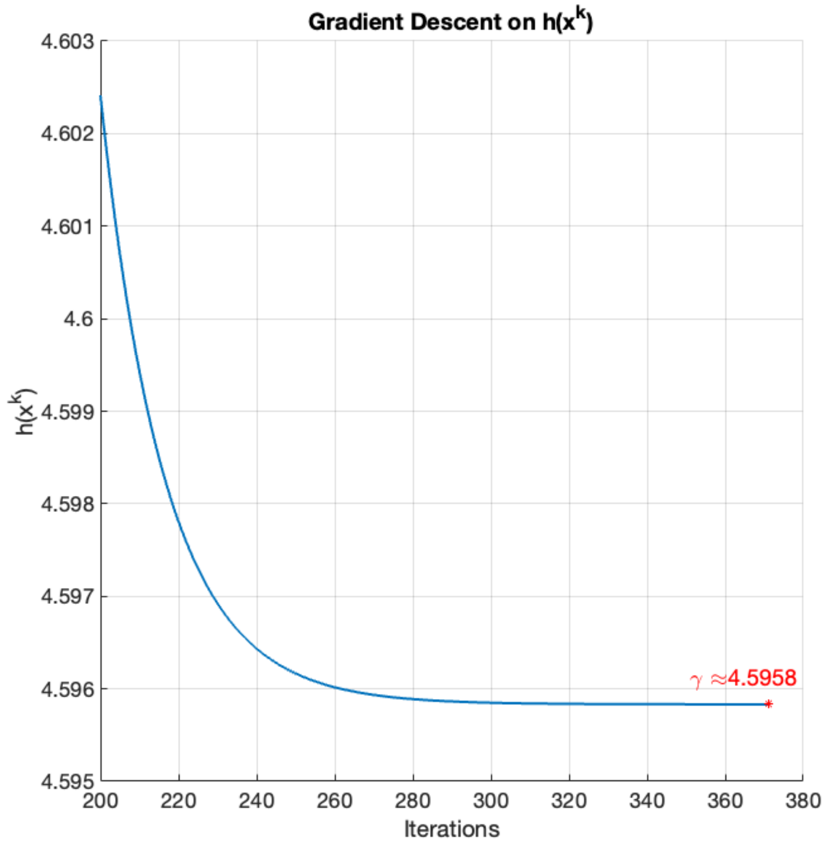

Figure 35: Gradient descent on (x) with $n = 6$ for last larger iterations

The next function family we check is in the form

$$\phi_2(x,y) = \begin{bmatrix} g(x)\cos(y) \\ u(x)\sin(y) \end{bmatrix} \tag{5.66}$$

where $g(x)$ and $u(x)$ are in the form described in Eq.(5.61) and Eq.(5.62).

Since a large n will result in the number of terms produced by the norms increasing drastically, leading to significantly long computation time, we start our algorithm with $n = 4$.

After running nearly 1100 iterations, we get the following results

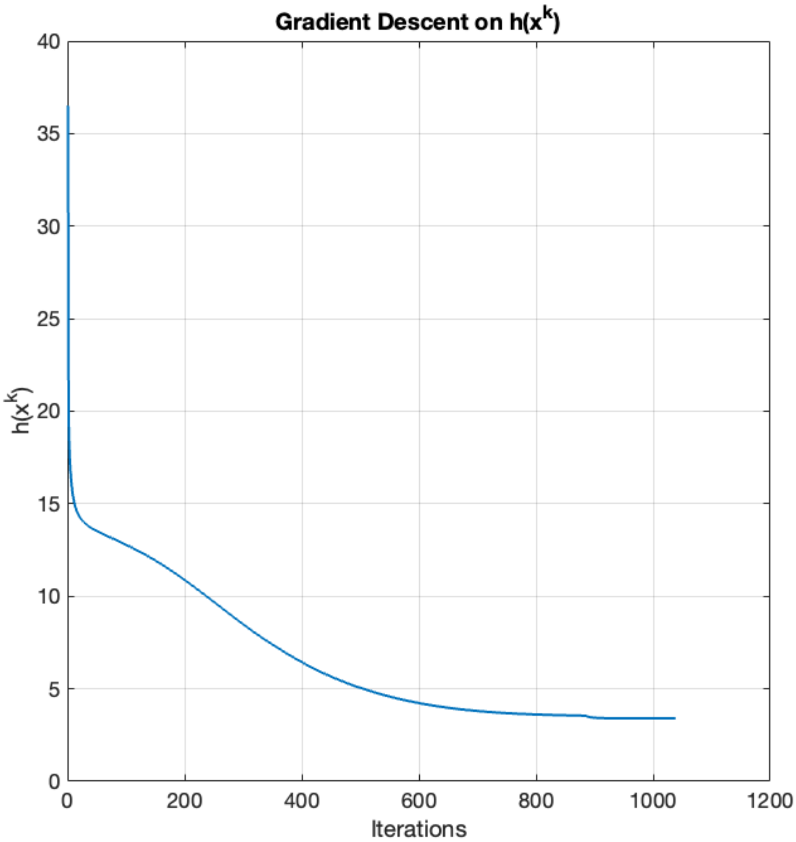

Figure 36: Gradient Descent on $h(x)$ with $n = 4$

The gradient descent on γ converges at $\gamma \approx 3.4128$.

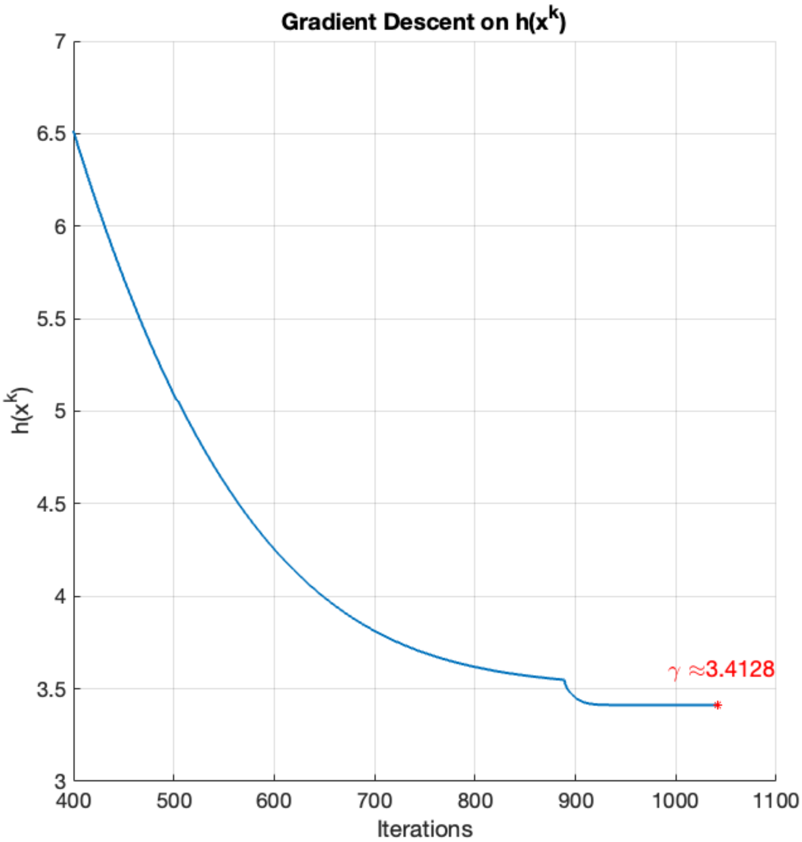

Figure 37: Gradient Descent on $h(x)$ with $n = 4$ for last larger iterations

With previous experience, we then increase n to 5.

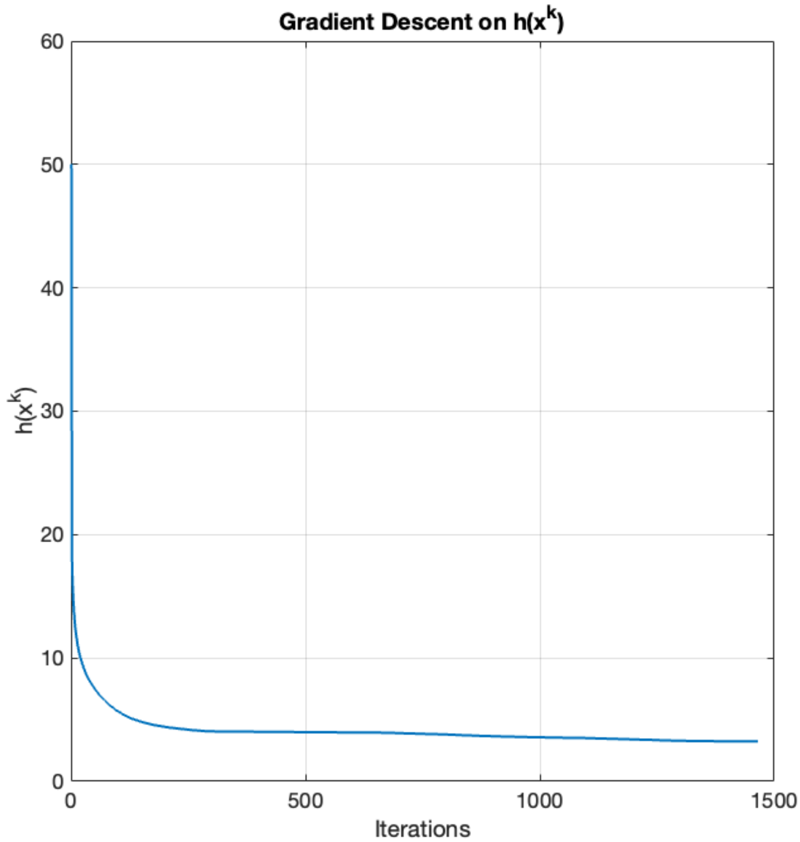

Figure 38: Gradient Descent on $h(x)$ with $n = 5$

It takes longer for the algorithm to converge. This time the smallest value we can get for γ is around 3.2578.

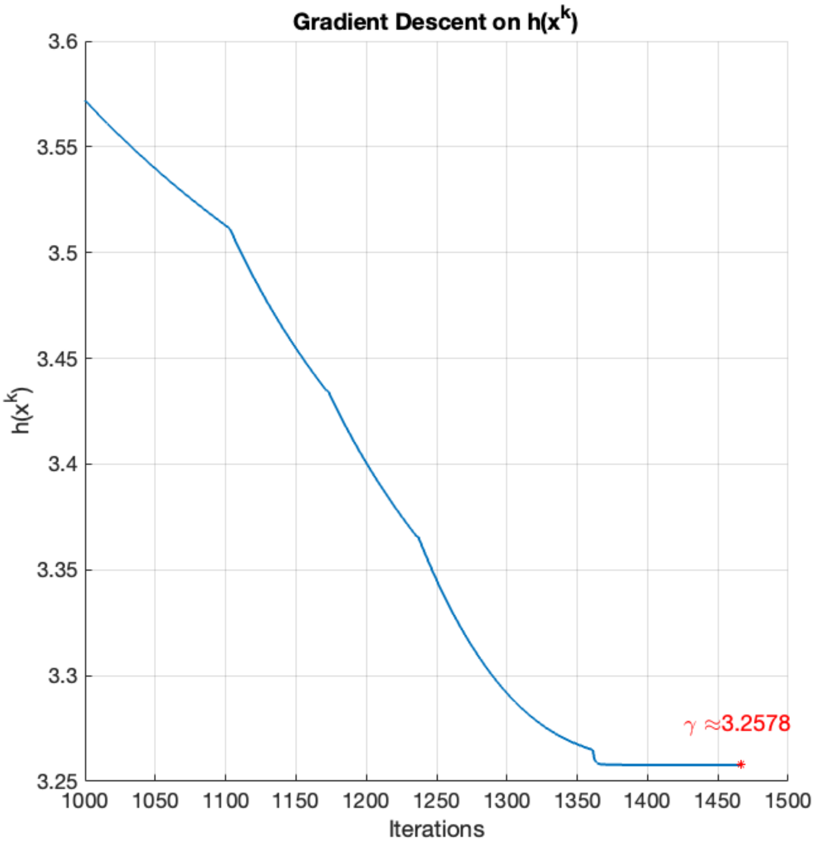

Figure 39: Gradient Descent on $h(x)$ with $n = 5$ for last larger iterations

The last function family we check is in the form

$$\phi_3 = \begin{bmatrix} g(x)h(y) \\ u(x)v(y) \end{bmatrix} \tag{5.67}$$

$g(x)$ and $u(x)$ are still in the form described in Eq.(5.61) and Eq.(5.62), we define them as

$$g(x) = a_0 + \sum_{i=1}^{n_1} a_i \sin(i\ x) + \sum_{j=1}^{n_1} b_j \cos(j\ x) \tag{5.68}$$

$$u(x) = c_0 + \sum_{i=1}^{n_2} c_i \sin(i\ x) + \sum_{j=1}^{n_2} d_j \cos(j\ x) \tag{5.69}$$

and

$$h(y) = e_0 + \sum_{i=1}^{n_3} e_i \sin(i\ y) + \sum_{j=1}^{n_3} f_j \cos(j\ y) \tag{5.70}$$

$$v(y) = g_0 + \sum_{i=1}^{n_4} g_i \sin(i\ y) + \sum_{j=1}^{n_4} h_j \cos(j\ y) \tag{5.71}$$

If we continue to set all n to be 4, there will be a total of 36 coefficients, resulting in extraordinarily long runtime. In order to obtain a relatively reasonable runtime, we start with $n_{1,2,3,4} = 1$.

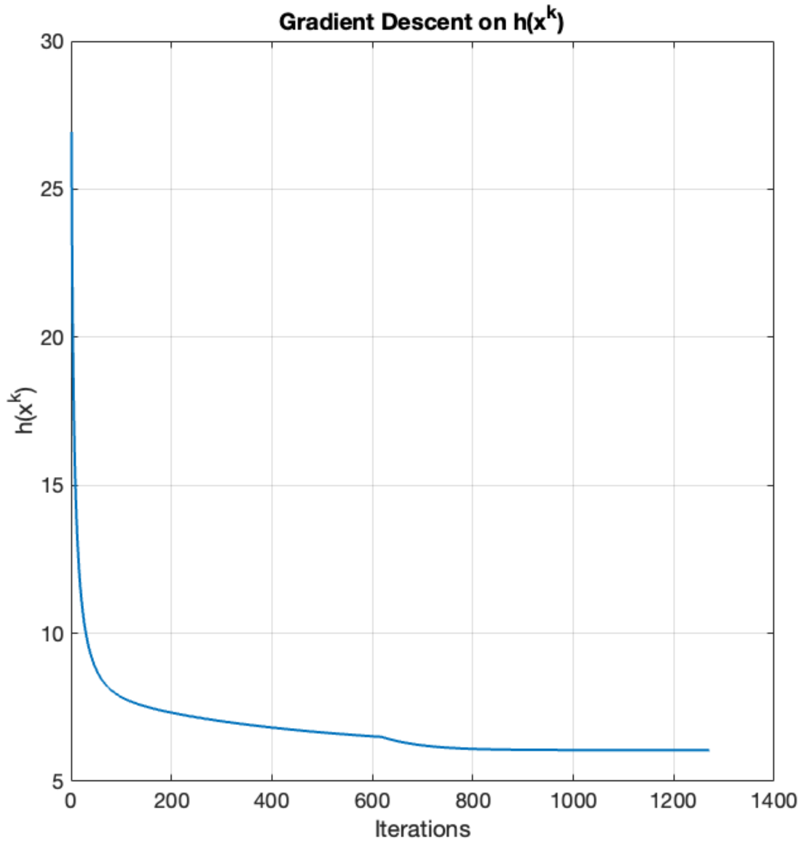

Figure 40: Gradient descent on $h(x)$ with $n_{1,2,3,4} = 1$

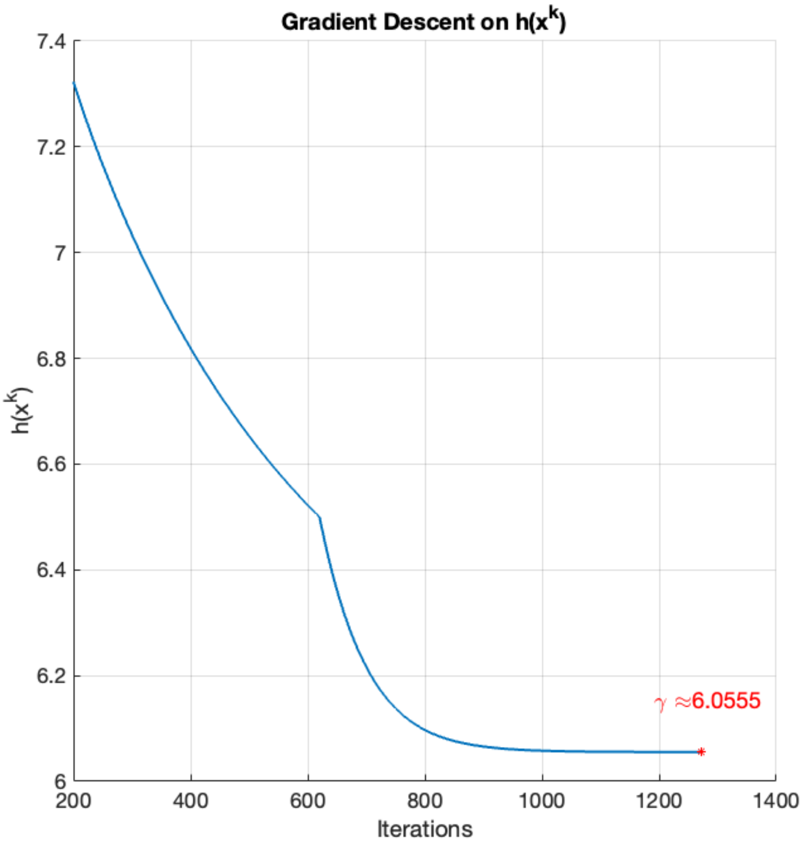

Figure 41: Gradient descent on $h(x)$ with $n_{1,2,3,4} = 1$ for last larger iterations

We gradually increase the number of coefficients and observe whether the minimal value of γ will decrease. Next we proceed with $n_3 = 2$, so

$$h(y) = e_0 + \sum_{i=1}^{2} e_i \sin(i\ y) + \sum_{j=1}^{2} f_j \cos(j\ y) \tag{5.72}$$

The results show that with 2 more coefficients, the best γ value we can get is reduced to 5.5468.

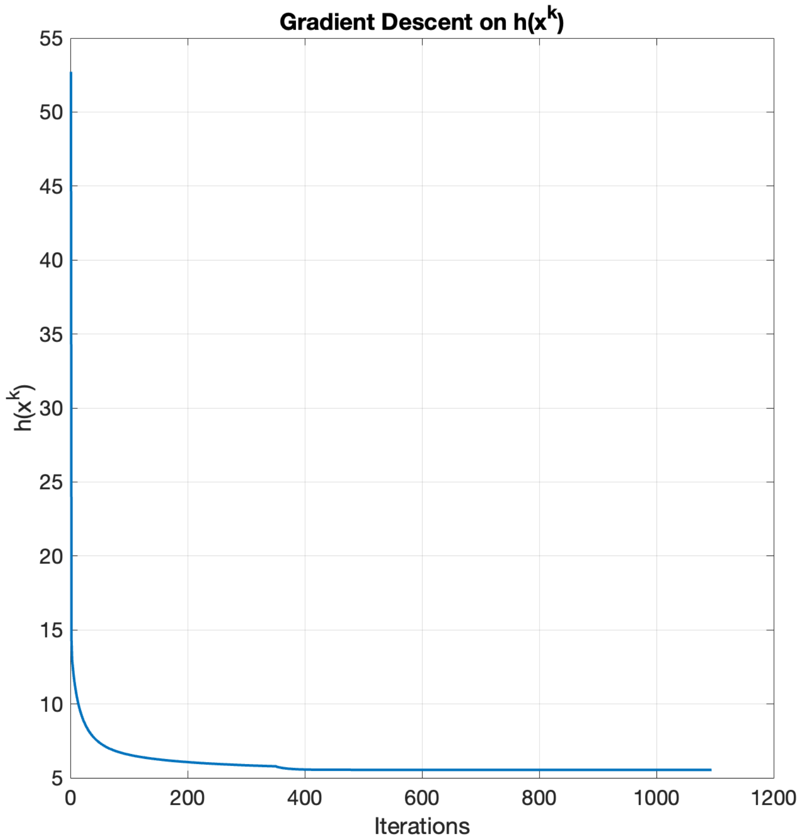

Figure 42: Gradient descent on $h(x)$ with $n_{1,2,4} = 1, n_3 = 2$

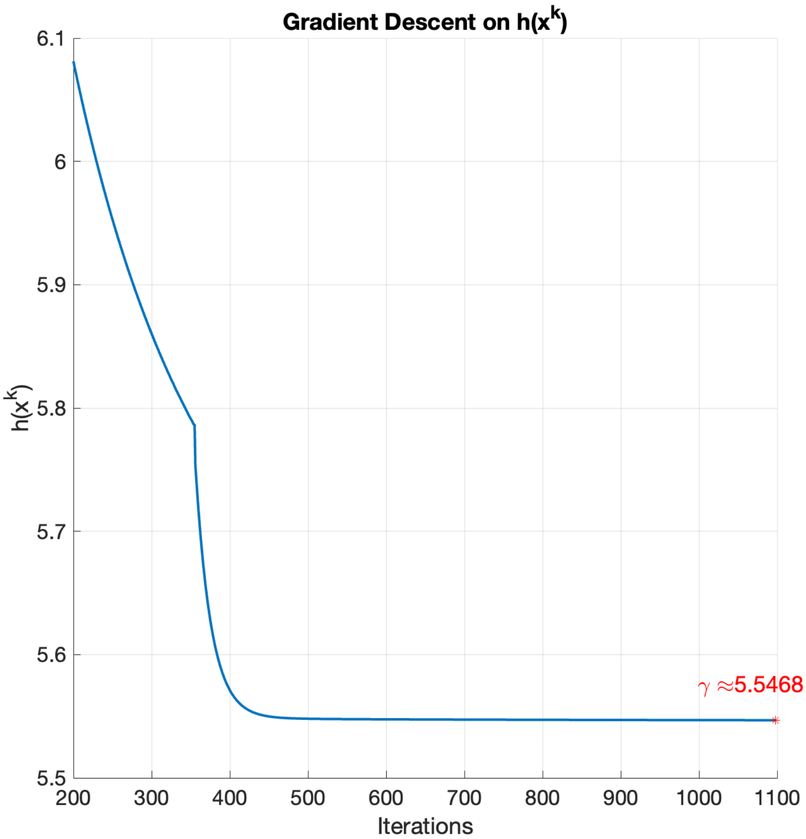

Figure 43: Gradient descent on $h(x)$ with $n_{1,2,4} = 1, n_3 = 2$ for last larger iterations)

We continue in this fashion and add 1 to n_4 so now there are 16 coefficients in total.

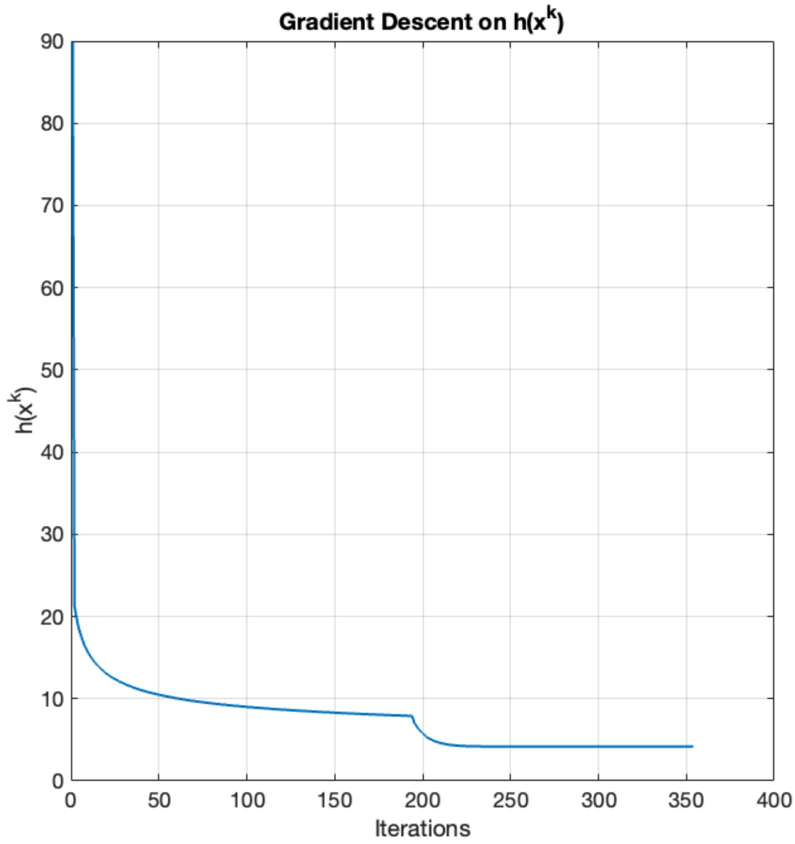

Figure 44: Gradient descent on $h(x)$ with $n_{1,2} = 1, n_{3,4} = 2$

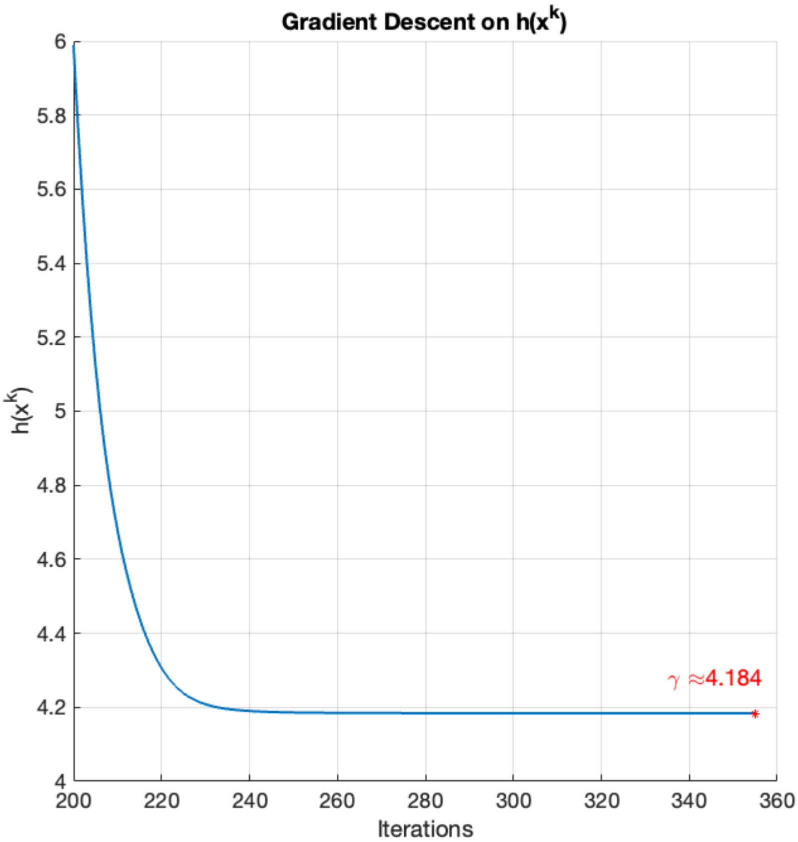

Figure 45: Gradient descent on $h(x)$ with $n_{1,2} = 1, n_{3,4} = 2$ for last larger iterations

6 Discussion

The results we found are of several types and demonstrate that Morrey's conjecture is valid, at least numerically speaking. Figure 4 - Figure 27 represent the changes of $d_\gamma(\phi^k) = J_\gamma(\phi^k) - f_\gamma(\xi)$ as the iterations proceed for various values of ξ and γ. The oscillations observed in Figure 24 - Figure 27 do not characterize any default of the algorithm we designed and used; they appear because we used random values of ξ: at each of the steepest descent algorithm iterations, a new value of the matrix ξ is used. Doing so allows us to increase our chance of finding a matrix ξ, a value of γ and a mapping ϕ for which the Jensen inequality is violated. And indeed, for instance, for the values of $\xi = \begin{bmatrix} 0.7246 & 0.01548 \\ 0.20019 & 0.82154 \end{bmatrix}$ and $\gamma = 2.2918$, our numerical simulations show that the expression Eq.(5.52) becomes negative, which violates the Jensen's inequality, and thereby confirming that the function is not quasi-convex. The rest of the analogous values of ξ and γ can be found from Table 1 - Table 4. The four figures Figure 24 - Figure 27 differ from each other by the initial trial functions used to enter the steepest descent algorithm. We obtained the violation of the Jensen's inequality for initial guess mappings that are

$$\begin{aligned}
\phi_1(x,y) &= \left[\sin(x(x-1)y(y-1)), \sin(x(x-1)y(y-1))^2\right] \\
\phi_2(x,y) &= \left[(x(x-1)y(y-1)), 0\right] \\
\phi_3(x,y) &= \left[\frac{1}{2\pi}\sin(2\pi x), \frac{1}{2\pi}\sin(2\pi y)\right] \\
\phi_4(x,y) &= \left[\frac{1}{100}\sin(\pi x), \frac{1}{100}\sin(\frac{3\pi}{2}y)\right].
\end{aligned}$$

For fixed values of ξ, the values of the Jensen function at each iteration shown in Figure 8 - Figure 14 show a decrease of the Jensen function until a minimal value after which it becomes a plateau or starts growing slowly. The oscillations observed in some of those figures maybe the results of numerical approximations that probably come from the discretization of the domain we used as it is the case in the numerical simulation of problem using finite element or finite difference methods. The decrease of the Jensen function is in the line of the steepest descent approach we used for the minimization problem we are solving. However, it is unclear the origin of the slow growth of this function observed in some of the figures after reaching a minimal point. It is also unclear why there is a sharp descent followed by a substantial growth.

Another investigation we have performed was to reproduce Duke University work results [10] by using a software we independently developed and exercised. Our code is based on the Duke University's suggestion of finding γ such that γ is bounded below by the right-hand side of the Eq.(5.58) for a given ϕ. Duke University suggests some families of functions ϕ, see for reference the functions ϕ defined in Section 5.3.1. We are looking for the one that realizes the supremum of the right-hand side of the Eq.(5.58) over these families of functions ϕ. If we do find this supremum γ such that f_γ is rank-one convex, then we would have found a function that is rank-one convex but not quasi-convex. We then changed this problem into a minimization problem on the coefficients in the functions defined in Section 5.3.1 and used a gradient descent algorithm to solve such problem using Maple software. The values of the coefficients that realize minimal of our version of the problem could be associated to the sought mapping ϕ. Figure 30 - Figure 45 represent the change of $h(x)$ in Eq.(5.60) during the iterations. The best value of γ we obtained is 3.2578 with

$$\phi(x,y) = \begin{bmatrix} g(x)\cos(y) \\ u(x)\sin(y) \end{bmatrix}$$

where

$$g(x) = a_0 + \sum_{i=1}^{5} a_i \sin(i\ x) + \sum_{j=1}^{5} b_j \cos(j\ x)$$

and

$$u(x) = c_0 + \sum_{i=1}^{5} c_i \sin(i\ x) + \sum_{j=1}^{5} d_j \cos(j\ x).$$

For this type of functions, Duke University found $\gamma = 3.91$. Even though the value of γ we found is not making the function f_γ rank-one convex, we have significantly improved Duke University's research results. We believe that this improvement is due to the refinement of the function ϕ we used. Indeed, our function ϕ differs from the one from Duke University by the fact that we augment the number of terms in $g(x)$ and $u(x)$.

For a function family of the form

$$\phi = \begin{bmatrix} g(x)h(y) \\ u(x)v(y) \end{bmatrix}$$

where $g(x), u(x), h(y)$ and $v(y)$ are defined at Eq.(5.68) and Eq.(5.70). Even though the Duke University's best result was obtained as $\gamma = 3.19$, our research shows that by increasing the number of terms in the trial functions, the values of γ are decreasing, see Figure 40 - Figure 45, until $\gamma = 4.184$, value after which due to computational limitation we could not keep the refinement of the trial function going. Our strong belief is that the more refine function we use, the closer we will get γ to the value that makes f_γ rank-one convex, but this is left for future work.

7 Conclusion Remarks

7.1 Conclusion of our work

We report here the results of some numerical simulations we performed to address the Morrey's conjecture [27] problem. Unlike in the method proposed by Dacorogna [13] and Gremaud [15], where the gradient descent algorithm was either "ad hoc" or unrelated to the minimization problem to be addressed, our approach used an exact vector gradient of the functional $I(\phi)$ to be minimized over a Sobolev space which we defined previously. We derived an exact expression of this gradient. Then, we solved the minimization problem numerically by using the approximated values of the trial functions at each of the nodes on the mesh we used. Once the updated values of the trial functions at the nodes are obtained, we used them to check the Jensen's inequality which is associated to the quasi-convexity property of the function f_γ. The initial trial functions to enter the gradient descent iterative algorithm are chosen as oscillating functions for some fixed values of the 2×2 matrix ξ. By randomizing the entries of ξ, we successfully used for each of the iterations a new value of ξ. For instance, the procedure gets us luck: for $\gamma = 2.2958$, that is f_γ rank-one convex, we found that the Jensen's inequality is violated (the function f_γ is not quasi-convex) for $\xi = \begin{bmatrix} 0.66684 & 0.03839 \\ 0.097573 & 0.65968 \end{bmatrix}$, and thereby validating Morrey's conjecture, at least numerically.

In addition, we also show the results of the improvements we performed on Duke University's [10] results for the same minimization problem. The gradient descent algorithm we developed and used demonstrated that by refining the trial functions suggested by Duke University, we obtained a value for $\gamma = 3.2578$ while Duke University's trial functions yield 3.91. Even though we were not successful in finding a value of γ for which f_γ is rank-one convex, we significantly reduced the value of γ than Duke University does. Many of our numerical calculations were stopped due to the limitations of our computational resources. We believe that with a more powerful computational infrastructure we can keep on improving Duke University's results with our algorithm. These efforts are left for future investigations.

7.2 Potential future works

Future research directions on this project consist of examining the connections between Morrey's conjecture with Iwaniec conjecture. Indeed, the Iwaniec conjecture is closely related to rank-one convexity and quasi-convexity properties, specifically to Morrey's and Šverák's conjectures. Note that if the Banuelos-Wang conjecture [8] is true, then the Iwaniec conjecture will be true. If the Banuelos-Wang conjecture is not true, then Morrey's conjecture would be settled for the case $n = m = 2$. The truth of the Iwaniec conjecture will impact the quasi-conformal mappings in $\mathbb{R}^n$. If the Iwaniec conjecture does hold, then it would be a stronger variation of Astala's area distortion theorem on quasi-conformal mappings, see Astala [3]. The truth of the Iwaniec conjecture would provide a mean to tell whether the Cauchy-Riemann operators ∂f, $\bar{\partial} f$

are similar to differentially subordinate harmonic functions, or differentially subordinate martingales. Experts believe that this is in fact the key to settling the Morrey's conjecture.

Other possible future research works on this project could consist of (i) developing robust numerical algorithms for the minimization problem at hand in the context of Morrey's conjecture problem, (ii) finding suitable Sobolev spaces on which the minimization problem will be performed, and (iii) discovering new mappings or improving the existing for the Sobolev space used in the minimization problem.

Software Availability

A version of the code developed for this work is available at: https://github.com/xdong99/Numerical-Quasiconvexity.

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