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Exact Tangent Stiffness Matrices for Ductile Damage Models of Gurson and its Extension to Include Strain Hardening Effects

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Contents

1. Introduction

The objective of this note is to expose the calculation of the stifness matrix for the classical model of ductile damage of Gurson [3], and its alternative due to Leblond-Perrin-Devaux (model LPD) [4] which have improved the modeling of the effects of work hardening in the model.

Perrin [4] presents the equations of the LPD model and its numerical implementation. Analogous presentation can be found for Gurson's model in Enakoutsa et al. [1] and Enakoutsa [2]. With regard to the latter, it will therefore be essential to expose in preliminary the basic equations of the model and its numerical implementation.

In fact, we will re-discuss certain aspects of the numerization of the two models. Indeed, the calculation of the stiffness matrix revealed a default of the numerical implementation proposed for the LPD model (and its analogous for the model of Gurson). This default concerns the use, in the flow rule associated with the criterion, of a porosity $f^{\frac{1}{2}}$ at the "half-interval", i.e. at time $t + \frac{\Delta t}{2}$, during the transition from time t to time $t + \Delta t$. The purpose of introducing this porosity was to improve the numerical precision of the algorithm. Unfortunately it presents the serious disadvantage of dissymmetrizing the stiffness matrix in the case of the model of Gurson (one will not seek here to justify this assertion, which would lead us too far). We prefer a slightly different algorithm using, instead of $f'^{\frac{1}{2}}$, the final porosity (at $t + \Delta t$) f. This leads, for Gurson model, to a symmetric matrix. For the LPD model, the matrix obtained will unfortunately be asymmetrical even by taking $f^{\frac{1}{2}} = f$; we will nevertheless favor this choice for the sake of homogeneity with the numerical implementation of Gurson model.

We will not in fact calculate all the terms of the tangent-matrix but only the "most important" ones (or at least that we think so). Thus one will not take into account for the calculation of the stiffness matrix, the variation of the stresses due to the variation of the temperature; this is in fact strictly licit due to the fact that this constraint correction is carried out explicitly, using the constraints at time t and not $t + \Delta t$, and is therefore independent of the displacement increment Δu between these times. We will not take into account either the variation of the stresses due to the objective derivation in the law of hypo-elasticity, which does indeed depend on Δu and therefore theoretically generates a contribution in the tangent-matrix.

Similarly, we will not take into account the influence of the geometry on the residual forces (derivation of $\mathbf{B}$ with respect to Δu in the integral $\int_\Omega^t \mathbf{B} : \boldsymbol{\sigma} \mathrm{dv}$). We can summarize all this by saying that the calculation of the stiffness matrix will be carried out by neglecting the effects of large transformations, that is to say by limiting ourselves to the calculation of $\partial\sigma_{ij}/\partial\Delta\varepsilon_{kl}$ where σ denote the stress tensor at $t + \Delta t$ and $\Delta\varepsilon$ the total strain increment (elastic + plastic) between t and $t + \Delta t$. This choice is in conformity with that already made in finite element codes for the computation of the stiffness matrix for the usual models of plasticity (without damage), whose numerical experiments demonstrated the effectiveness.

2. TANGENT STIFFNESS MATRIX FOR GURSON MODEL

2.1. RECALL OF THE EQUATIONS OF THE MODEL

As already mentioned in the Introduction, the models of Gurson and LPD differ only with regard to the modeling of work hardening. In Gurson's model, the parameters σ_1 and σ_2 of the LPD model are reduced to a single parameter $\bar{\sigma}$ (which represents a kind of average elastic limit of the metallic matrix) , both in the plasticity criterion and the associated flow rule This criterion and this flow rule are written as follows:

$$\frac{\sigma_{eq}^2}{\bar{\sigma}^2} + 2p \operatorname{ch}\left(\frac{3}{2}\frac{\sigma_m}{\bar{\sigma}}\right) - 1 - p^2 = 0 \tag{1}$$

$$\begin{cases} \dot{\varepsilon}^{p'} = \dfrac{3}{2}\dfrac{\dot{\varepsilon}_{eq}}{\sigma_{eq}}\sigma' \\ \dfrac{\dot{\varepsilon}_m^p}{\dot{\varepsilon}_{eq}^p} = \dfrac{p}{2}\dfrac{\bar{\sigma}}{\sigma_{eq}}\operatorname{sh}\dfrac{3}{2}\dfrac{\sigma_m}{\bar{\sigma}} \end{cases} \tag{2}$$

where the symbol $'$ denotes the deviatoric part, and where we recall that:

$$p = qf^*, f^* = \begin{cases} f & \text{si } f \leq f_c \\ f_c + \delta\,(f - f_c) & \text{si } f > f_c. \end{cases} \tag{3}$$

(f denoting the porosity, q the "Tvergaard parameter" [6] and f_σ, δ the "Needleman-Tvergaard [5] coalescence parameters") .

The parameter $\bar{\sigma}$ is expressed according to the "mean equivalent deformation" $\bar{\varepsilon}$ by the relation:

$$\bar{\sigma} = \sigma(\bar{\varepsilon}) \tag{4}$$

where $\sigma(\varepsilon)$ represents the function giving the elastic limit in simple tension of the sound metal matrix according to the cumulated equivalent deformation. The evolution law of the parameter $\bar{\sigma}$ is written:

$$(1 - f)\bar{\sigma}\dot{\bar{\varepsilon}} = \sigma : \dot{\varepsilon}^p = \sigma_{eq}\dot{\varepsilon}_{eq}^p + 3\sigma_m\dot{\varepsilon}_m^p. \tag{5}$$

2.2. NUMERICAL IMPLEMENTATION - ESSENTIAL ELEMENTS

The basic numerical equations, obtained by replacing σ_1 and σ_2 by σ in the formulas of [4] and by discretizing in time the evolution equation Eq.(5) , are written, noting with an exponent $^\circ$ the quantities taken at t and without a special symbol those taken at $t + \Delta t$:

2.2.1. Partition of the increment of the total deformation and elasticity law

:

$$\begin{cases} \sigma' = \sigma^{*'} - 2\mu\Delta\varepsilon^{p'}, \sigma^{*'} = \sigma^{o'} + 2\mu\Delta\varepsilon' \\ \sigma_m = \sigma_m^* - (3\lambda + 2\mu)\Delta\varepsilon_m^p, \sigma_m = \sigma_m^o + (3\lambda + 2\mu)\Delta\varepsilon_m \end{cases} \tag{6}$$

(σ represents the stress tensor at $t + \Delta t$ "computed elastically")

2.2.2. Parametrization of the yield criterion

$$\begin{cases} \sigma_{eq} = (1 - p)\bar{\sigma}\cos\varphi \\ \sigma_m = \frac{2}{3}\bar{\sigma}\operatorname{sgn}(\varphi)\operatorname{Argch}\left[1 + \frac{(1-p)^2}{2p}\sin^2\varphi\right], \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{cases} \tag{7}$$

2.2.3. Flow rule

:

$$\begin{cases} \Delta\varepsilon^{p'} = \dfrac{3}{2}\dfrac{\Delta e^{p}_{eq}}{\sigma_{eq}}\sigma' \\ \dfrac{\Delta e^{p}_{m}}{\Delta\varepsilon^{p}_{eq}} = \dfrac{p}{2}\dfrac{\bar{\sigma}}{\sigma_{eq}}\,\mathrm{sh}\left(\dfrac{3}{2}\dfrac{\sigma_m}{\bar{\sigma}}\right) \end{cases} \tag{8}$$

2.2.4. Law of evolution of equation of $\bar{\varepsilon}$

:

$$(1-f)\bar{\sigma}\Delta\bar{\varepsilon} = \sigma_{eq}\Delta\varepsilon^{p}_{eq} + 3\sigma_m\Delta\varepsilon^{p}_{m} \tag{9}$$

Note the essentially implicit character of this algorithm; thus the stress intervening in the flow rule are those of the time $t + \Delta t$. the same is true for the parameter $\bar{\sigma}$. (Adopting an explicit algorithm with respect to this parameter would mean treating the problem between t and $t + \Delta t$ as in perfect plasticity, which would inevitably lead to a singular tangent matrix). On the other hand, it is pointed out that the algorithm is explicit with respect to the porosity; f above does not in fact designate the true porosity at $t + \Delta t$, unknown, but an approximation of this porosity, known and fixed during the transition from t to $t + \Delta t$, given by:

$$f = f^o + f^o\Delta t \tag{10}$$

(One can show that the use of an implicit algorithm with respect to f would lead to an asymmetric stiffness matrix).

The processing of these equations leads to the following results.
First of all,

$$\begin{cases} \Delta\varepsilon^{p}_{eq} = \left(\frac{\sigma^{*}_{eq}-\sigma_{eq}}{3\mu}\right); \quad \sigma' = \frac{\sigma_{eq}}{\sigma_{eq}}\sigma^{*\prime} \\ \Delta\varepsilon^{p}_{m} = \frac{\sigma^{*}_{m}-\sigma_{m}}{3\lambda+2\mu} \end{cases} \tag{11}$$

These equations bring back the calculation of the unknowns $\sigma', \sigma_m, \Delta\varepsilon^{p}_{eq}, \Delta\varepsilon^{p}_{m}$ to that of the only quantities σ_{eq} and σ_m. Given the parametrization Eq.(7), there remains only 2 unknowns, φ and $\bar{\sigma}$ (or $\Delta\bar{\varepsilon}$). The flow rule Eq.(8)$_2$ further leads to the equation:

$$F = a\,(\sigma^{*}_{m} - \sigma_m)\cos\varphi - p\left(\sigma^{*}_{eq} - \sigma_{eq}\right)\mathrm{sh}\left(\frac{3}{2}\frac{\sigma_m}{\sigma}\right) = 0, \quad a = (1-p)\frac{6\mu}{3\lambda + 2\mu} \tag{12}$$

which can be solved with respect to φ, at fixed $\bar{\sigma}$, by Newton's method. The other unknowns are deduced from it by the previous equations. The evolution equation Eq.(9) makes it possible to "update" $\Delta\bar{\varepsilon}$, therefore $\bar{\sigma}$, and therefore to calculate this unknown by an iterative method of fixed point (φ being calculated by Newton's method at each iteration).

2.3. CALCULATION OF THE STIFFNESS MATRIX

2.3.1. New parametrization of the criterion and derivation

The quantities σ_{eq}, σ_m are expressed as a function of φ and $\bar{\sigma}$ as follows (see Eq.(7):

$$\begin{cases} \sigma_{eq} = \bar{\sigma} S_{eq}, S_{eq} \equiv S_{eq}(\varphi) = (1-p)\cos\varphi \\ \sigma_m = \bar{\sigma} S_m, S_m \equiv S_m(\varphi) = \frac{2}{3}\,\mathrm{sgn}(\varphi)\,\mathrm{Argch}\left[1 + \frac{(1-p)^2}{2p}\sin^2\varphi\right] \end{cases} \tag{13}$$

the derivatives of S_{eq} and S_m with respect to φ being given by

$$\begin{cases} \frac{dS_{eq}}{d\varphi} = -(1-p)\sin\varphi \\ \frac{dS_m}{d\varphi} = \frac{2}{3}\frac{(1-p)^2}{p}\frac{\sin\varphi\cos\varphi}{\mathrm{sh}\left(\frac{3}{2}S_m\right)} \end{cases} \tag{14}$$

2.3.2. Derivatives of $\sigma^{*\prime}, \sigma^*_m$ and σ^*_{eq} with respect to $\Delta\varepsilon$

From formulas

$$\sigma^{*\prime}_{ij} = \sigma^{o'}_{ij} + 2\mu\Delta\varepsilon'_{ij}, \Delta\varepsilon'_{ij} = \Delta\varepsilon_{ij} - \frac{1}{3}\Delta\varepsilon_{kk}\delta_{ij}, \quad \sigma^*_m = \sigma^o_m + (3\lambda+2\mu)\Delta\varepsilon_m = \sigma^o_m + \frac{1}{3}(3\lambda+2\mu)\Delta\varepsilon_{kk}, \tag{15}$$

we draw immediately:

$$\begin{cases} \frac{\partial\sigma^{*\prime}_{ij}}{\partial\Delta\varepsilon_{kl}} = 2\mu\left[\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{1}{3}\delta_{ij}\delta_{kl}\right] \\ \frac{\partial\sigma^*_m}{\partial\Delta\varepsilon_{kl}} = \frac{1}{3}(3\lambda+2\mu)\delta_{kl} \end{cases} \tag{16}$$

From Eq.(16), we deduce that:

$$\frac{\partial\sigma^*_{eq}}{\partial\Delta\varepsilon_{kl}} = 3\mu\frac{\sigma^{*\prime}_{kl}}{\sigma^*_{eq}} \tag{17}$$

2.3.3. *Derivatives of $\bar{\sigma}$ with respect to $\Delta\varepsilon$ and φ*

To evaluate these derivatives, we will differentiate the evolution equation Eq.(9) from $\Delta\bar{\varepsilon}$, written in the form:

$$(1-f)\Delta\bar{\varepsilon} = \mathrm{S_{eq}}\Delta\varepsilon^{\mathrm{p}}_{\mathrm{eq}} + 3\mathrm{S_m}\Delta\varepsilon^{\mathrm{p}}_{\mathrm{m}} = \mathrm{S_{eq}}\frac{\sigma^*_{\mathrm{eq}} - \bar{\sigma}\mathrm{S_{eq}}}{3\mu} + 3\mathrm{S_m}\frac{\sigma^*_{\mathrm{m}} - \bar{\sigma}S_{\mathrm{m}}}{3\lambda + 2\mu} \tag{18}$$

Before taking this differentiation, note that:

$$\frac{d\mathrm{S_{eq}}}{\mathrm{d}\varphi}\Delta\varepsilon^{\mathrm{p}}_{\mathrm{eq}} + 3\frac{\mathrm{dS_m}}{\mathrm{d}\varphi}\Delta\varepsilon^{\mathrm{p}}_{\mathrm{m}} = 0; \tag{19}$$

this property is due to the orthogonality of $\Delta\varepsilon^{\mathrm{p}}$ to the yield surface ($\sigma : \Delta\varepsilon^{\mathrm{p}} = \mathrm{d}\sigma_{\mathrm{eq}}\Delta\varepsilon^{\mathrm{p}}_{\mathrm{eq}} + 3\mathrm{d}\sigma_{\mathrm{m}}\Delta \mathrm{c}^{\mathrm{p}}_{\mathrm{m}} = 0$ if we vary σ on the yield surface, i.e. if we vary φ, at $\bar{\sigma}$ fixed). Given this remark, the differentiation gives:

$$\begin{aligned}(1-\mathrm{f})\mathrm{d}\Delta\bar{\mathrm{c}} &= \left(\frac{\mathrm{S_{eq}}}{3\mu}\frac{\partial\sigma^*_{\mathrm{eq}}}{\partial\Delta \mathrm{e_{kj}}} + \frac{3\mathrm{S_m}}{3\lambda+2\mu}\frac{\partial\sigma^*_{\mathrm{m}}}{\partial\Delta\varepsilon_{\mathrm{kl}}}\right)\mathrm{d}\Delta\varepsilon_{\mathrm{kl}} \\ &\quad -\left(\frac{\sigma_{\mathrm{eq}}}{3\mu}\frac{\mathrm{dS_{eq}}}{\mathrm{d}\phi} + \frac{3\sigma_{\mathrm{m}}}{3\lambda+2\mu}\frac{\mathrm{dS_m}}{\mathrm{d}\phi}\right)\mathrm{d}\phi - \left(\frac{\mathrm{S^2_{eq}}}{3\mu} + \frac{3\mathrm{S^2_m}}{3\lambda+2\mu}\right)\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\bar{\varepsilon}}\mathrm{d}\Delta\bar{\varepsilon}\end{aligned} \tag{20}$$

At $\varphi = cst$, this gives:

$$\frac{\partial\bar{\varepsilon}}{\partial\Delta\varepsilon_{\mathrm{kj}}} = \frac{\frac{\mathrm{S_{eq}}}{3\mu}\frac{\partial\sigma^*_{\mathrm{eq}}}{\partial\Delta \mathrm{c_{kj}}} + \frac{3\mathrm{S_m}}{3\lambda+2\mu}\frac{\partial\sigma^*_{\mathrm{m}}}{\partial\Delta \mathrm{r_{kl}}}}{(1-\mathrm{f}) + \left(\frac{\mathrm{S^2_{eq}}}{3\mu} + \frac{3\mathrm{S^2_m}}{3\lambda+2\mu}\right)\frac{\mathrm{d}\bar{\sigma}}{\mathrm{de}}} \quad \left(and \quad \frac{\partial\bar{\sigma}}{\partial\Delta\varepsilon_{\mathrm{kl}}} = \frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\bar{\varepsilon}}\frac{\partial\Delta\bar{\varepsilon}}{\partial\Delta\varepsilon_{\mathrm{kl}}}\right); \tag{21}$$

at $\Delta\varepsilon = cst$, we get:

$$\frac{\partial\Delta\bar{\varepsilon}}{\partial\varphi} = -\frac{\frac{\sigma_{\mathrm{eq}}}{3\mu}\frac{\mathrm{dS_{eq}}}{\mathrm{d}\varphi} + \frac{3\sigma_{\mathrm{m}}}{3\lambda+2\mu}\frac{\mathrm{dS_m}}{\mathrm{d}\varphi}}{(1-\mathrm{f}) + \left(\frac{\mathrm{S^2_{eq}}}{3\mu} + \frac{3\mathrm{S^2_m}}{3\lambda+2\mu}\right)\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\bar{\varepsilon}}} \quad \left(and \quad \frac{\partial\bar{\sigma}}{\partial\varphi} = \frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\bar{\varepsilon}}\frac{\partial\bar{\varepsilon}}{\partial\varphi}\right) \tag{22}$$

2.3.4. *Derivatives of φ with respect to $\Delta\varepsilon$*

It is now necessary to exploit the fundamental equation Eq.(12) giving φ. Note that the stresses $\sigma_{\mathrm{eq}^*}, \sigma^*_{\mathrm{m}}$ depend only on $\Delta\varepsilon$, while the constraints $\sigma_{\mathrm{eq}}, \sigma_{\mathrm{m}}$ depend on φ and $\bar{\sigma}$. So this equation is written as:

$$F(\Delta\varepsilon, \varphi, \bar{\sigma}) = a\left[\sigma^*_{\mathrm{m}}(\Delta\varepsilon) - \bar{\sigma}\mathrm{S_m}(\varphi)\right]\cos\varphi - \mathrm{p}\left[\sigma^*_{\mathrm{eq}}(\Delta\varepsilon) - \bar{\sigma}\mathrm{S_{eq}}(\varphi)\right]\mathrm{sh}\left[\frac{3}{2}\mathrm{S_m}(\varphi)\right] = 0 \tag{23}$$

By differentiating, we get

$$\frac{\partial\mathrm{F}}{\partial\Delta\varepsilon_{\mathrm{kl}}}\mathrm{d}\Delta\varepsilon_{\mathrm{kl}} + \frac{\partial\mathrm{F}}{\partial\varphi}\mathrm{d}\varphi + \frac{\partial\mathrm{F}}{\partial\bar{\sigma}}\mathrm{d}\bar{\sigma} = 0. \tag{24}$$

where

$$\frac{\partial F}{\partial \Delta\varepsilon_{kl}} = \frac{\partial F}{\partial \sigma^*_{eq}} \frac{\partial \sigma^*_{eq}}{\partial \Delta\varepsilon_{kl}} + \frac{\partial F}{\partial \sigma^*_m} \frac{\partial \sigma^*_m}{\partial \Delta\varepsilon_{kl}}, \quad \frac{\partial F}{\partial \sigma^*_{eq}} = -p\,\mathrm{sh}\left(\frac{3}{2}S_m\right), \quad \frac{\partial F}{\partial \sigma^*_m} = a\cos\varphi, \tag{25}$$

$$\frac{\partial F}{\partial \varphi} = p\bar{\sigma}\mathrm{sh}\left(\frac{3}{2}S_m\right)\frac{dS_{eq}}{d\varphi} - \left[a\bar{\sigma}\cos\varphi + \frac{3}{2}p\left(\sigma^*_{eq} - \sigma_{eq}\right)\mathrm{ch}\left(\frac{3}{2}S_m\right)\right]\frac{dS_m}{d\varphi} - a\left(\sigma^*_m - \sigma_m\right)\sin\varphi \tag{26}$$

$$\frac{\partial F}{\partial \bar{\sigma}} = -aS_m\cos\varphi + pS_{eq}\mathrm{sh}\left(\frac{3}{2}S_m\right) \tag{27}$$

By expanding $d\bar{\sigma} = \frac{\partial \bar{\sigma}}{\partial \Delta\varepsilon_{kl}} d\Delta\varepsilon_k + \frac{\partial \bar{\sigma}}{\partial \varphi} d\varphi$ in Eq.(24), we get:

$$\frac{\partial F}{\partial \Delta\varepsilon_{kl}} d\Delta\varepsilon_{kl} + \frac{\partial F}{\partial \varphi} d\varphi + \frac{\partial F}{\partial \sigma}\frac{\partial \sigma}{\partial \Delta\varepsilon_{kl}} d\Delta\varepsilon_k + \frac{\partial F}{\partial \sigma}\frac{\partial \sigma}{\partial \varphi} d\varphi = 0 \tag{28}$$

and thus

$$\frac{\partial \varphi}{\partial \Delta\varepsilon_{kl}} = -\frac{\frac{\partial F}{\partial \Delta\varepsilon_{kl}} + \frac{\partial F}{\partial \bar{\sigma}}\frac{\partial \bar{\sigma}}{\partial \Delta\varepsilon_{kl}}}{\frac{\partial F}{\partial \varphi} + \frac{\partial F}{\partial \bar{\sigma}}\frac{\partial \bar{\sigma}}{\partial \varphi}}. \tag{29}$$

Note that the derivative $\partial F/\partial\varphi$ is precisely the "slope" which intervenes in Newton's method on φ used to solve equation Eq.(28).

2.3.5. Derivatives of σ_{eq} and σ_m with respect to $\Delta\varepsilon$

We have $\sigma_{eq} = \bar{\sigma}_{eq}, \bar{\sigma}$ depending on $\Delta\varepsilon$ and φ, and S_{eq} of φ alone, φ being itself a function of $\Delta\varepsilon$. Therefore:

$$\frac{\partial \sigma_{eq}}{\partial \Delta\varepsilon_{kl}} = S_{eq}\frac{\partial \bar{\sigma}}{\partial \Delta\varepsilon_{kl}} + S_{eq}\frac{\partial \bar{\sigma}}{\partial \varphi}\frac{\partial \varphi}{\partial \Delta\varepsilon_{kl}} + \bar{\sigma}\frac{dS_{eq}}{d\varphi}\frac{\partial \varphi}{\partial \Delta\varepsilon_{kl}} \tag{30}$$

Also,

$$\frac{\partial \sigma_m}{\partial \Delta\varepsilon_{kl}} = S_m\frac{\partial \bar{\sigma}}{\partial \Delta\varepsilon_{kl}} + S_m\frac{\partial \bar{\sigma}}{\partial \varphi}\frac{\partial \varphi}{\partial \Delta\varepsilon_{kl}} + \bar{\sigma}\frac{S_m}{d\varphi}\frac{\partial \varphi}{\partial \Delta\varepsilon_{kl}} \tag{31}$$

2.3.6. Derivatives of σ' and σ with respect to $\Delta\varepsilon$

The differentiation of the equation $\sigma'_{ij} = \frac{\sigma_{eq}}{\sigma^*_{eq}}\sigma^{*\prime}_{ij}$ gives

$$\frac{\partial \sigma^*_{ij}}{\partial \Delta\varepsilon_{kl}} = \frac{\sigma_{eq}}{\sigma^*_{eq}}\frac{\partial \sigma^*_{ij}}{\partial \Delta\varepsilon_{kl}} + \sigma^*_{ij}\left(\frac{1}{\sigma^*_{eq}}\frac{\partial \sigma_{eq}}{\partial \Delta\varepsilon_k} - \frac{\sigma_{eq}}{\sigma^{*2}_{eq}}\frac{\partial \sigma^*_{eq}}{\partial \Delta\varepsilon_{kl}}\right) \tag{32}$$

Finally, the equation $\sigma_{ij} = \sigma'_{ij} + \sigma_m\delta_{ij}$ gives

$$\frac{\partial \sigma_{ij}}{\partial \Delta\varepsilon_{kl}} = \frac{\partial \sigma'_{ij}}{\partial \Delta\varepsilon_{kl}} + \frac{\partial \sigma_m}{\partial \Delta\varepsilon_{kl}}\delta_{ij} \tag{33}$$

This completes the calculation of the tangent-matrix for the Gurson model.

3. STIFFNESS MATRIX FOR THE LPD MODEL

3.1. NUMERICAL EQUATIONS FOR THE LPD MODEL

The numerical equations of the LPD model having been exposed in [1], there is no need to re-expose it here. We will content ourselves with indicating the equations which are modified due to the replacement of the porosity $f^{(1/2)}$ at the half interval by (the explicit approximation of) the final porosity f.

The "mean part" of the discretized flow rule is now read:

$$\frac{\Delta\varepsilon_m^p}{\Delta\varepsilon_{\mathrm{eq}}^p} = \frac{p}{2}\frac{\sigma_1^2}{\sigma_2\sigma_{\mathrm{eq}}}\,\mathrm{sh}\left(\frac{3}{2}\frac{\sigma_{\mathrm{m}}}{\sigma_2}\right) \tag{34}$$

Moreover, the fundamental equation of the LPD model giving the value of φ is written:

$$\mathrm{F} = \mathrm{a}\frac{\sigma_2}{\sigma_1}\left(\sigma_{\mathrm{m}}^* - \sigma_{\mathrm{m}}\right)\cos\varphi - \mathrm{p}\left(\sigma_{\mathrm{eq}}^* - \sigma_{\mathrm{eq}}\right)\mathrm{sh}\left(\frac{3}{2}\frac{\sigma_{\mathrm{m}}}{\sigma_2}\right) = 0 \tag{35}$$

where a is given by the same formula as for the Gurson model, i.e. by the equation Eq.(12 $)_2$ of the present note . The purpose of writing the quantity "a" in the present form "$\mathrm{a}\frac{\sigma_2}{\sigma_1}$ " is to clearly show the dependence of this quantity with respect to the parameters σ_1 and σ_2.

Note finally a small difference in notation compared to Gurson model. For the latter, $\Delta\varepsilon_m$ represented above the average total deformation increment, whereas here it represents the increment of the hardening parameter ε_{m}, defined by law $\dot{\varepsilon}_{\mathrm{m}} = \left|\dot{\varepsilon}_{\mathrm{m}}^{\mathrm{p}}\right|$.

3.2. CALCULATION OF THE TANGENT STIFFNESS MATRIX

3.2.1. Yield locus parametrization

The stress $\sigma_{\mathrm{eq}}, \sigma_{\mathrm{m}}$ are expressed here as a function of φ, σ_1 and σ_2 as follows:

$$\begin{cases} \sigma_{eq} = \sigma_1 \mathrm{S}_{\mathrm{eq}}(\varphi) \\ \\ \sigma_{\mathrm{m}} = \sigma_2 \mathrm{S}_{\mathrm{m}}(\varphi) \end{cases} \tag{36}$$

where the expressions for $\mathrm{S}_{\mathrm{eq}}(\varphi)$ and $S_{\mathrm{m}}(\varphi)$, as well as their derivatives, are the same as for the Gurson's model (Eq.(13) and Eq.(14))

3.2.2. Differentials of $\varepsilon_{\mathrm{eq}}$ and σ_{m}

From the equation of evolution of the hardening parameter ε_{eq}, i.e. $\dot{\varepsilon}_{\mathrm{eq}} = \dot{\varepsilon}_{\mathrm{eq}}^{\mathrm{p}}$, we deduce that

$$\varepsilon_{\mathrm{eq}} = \varepsilon_{\mathrm{eq}}^{\mathrm{o}} + \Delta\varepsilon_{\mathrm{eq}}^{\mathrm{p}} \Rightarrow \mathrm{d}\varepsilon_{\mathrm{eq}} = \mathrm{d}\Delta\varepsilon_{\mathrm{eq}}^{\mathrm{p}} \tag{37}$$

Similarly, from the evolution equation $\dot{\varepsilon}_{\mathrm{m}} = |\dot{\varepsilon}_{\mathrm{m}}|$, we deduce that:

$$\varepsilon_{\mathrm{m}} = \varepsilon_{\mathrm{m}}^* + \left|\Delta\varepsilon_{\mathrm{m}}^{\mathrm{p}}\right| \Rightarrow \mathrm{d}\varepsilon_{\mathrm{m}} = \mathrm{sgn}\left(\Delta\varepsilon_{\mathrm{m}}^{\mathrm{p}}\right)\mathrm{d}\Delta\varepsilon_{\mathrm{m}}^{\mathrm{p}} \tag{38}$$

Now, according to Eq.(34), the sign of $\Delta\varepsilon_m^p$ is the same as that of σ_m, that is to say of φ according to Eq.(36 $)_2$ and the expression of $S_m(\varphi)$ (cf. Eq.(13)). Therefore

$$d\varepsilon_m = \varepsilon d\Delta\varepsilon_m^p, \quad \varepsilon = \text{sgn}(\varphi) \tag{39}$$

3.2.3. Derivatives of σ_1 and σ_2 with respect to $\Delta\varepsilon$ and φ

Let us recall that σ_1 and σ_2 are known, pre-tabulated functions of the hardening parameters $\varepsilon_{eq}, \varepsilon_m$ (and of the initial porosity). We have:

$$d\sigma_1 = \frac{\partial\sigma_1}{\partial\varepsilon_{eq}}d\varepsilon_{eq} + \frac{\partial\sigma_1}{\partial\varepsilon_m}d\varepsilon_m = \frac{\partial\sigma_1}{\partial\varepsilon_{eq}}d\Delta\varepsilon_{eq} + \frac{\partial\sigma_1}{\partial\varepsilon_m}\varepsilon d\Delta\varepsilon_m^p \tag{40}$$

from Eq.(37) and Eq.(39). Now,

$$\Delta\varepsilon_{eq}^p = \frac{\sigma_{eq}^* - \sigma_{eq}}{3\mu} = \frac{1}{3\mu}\left(\sigma_{eq}^* - \sigma_1 S_{eq}\right) \Rightarrow d\Delta\varepsilon_{eq}^p = \frac{1}{3\mu}\left(\frac{\partial\sigma_{eq}^*}{\partial\Delta\varepsilon_{kl}}d\Delta\varepsilon_{kd} - \sigma_1\frac{dS_{eq}}{d\varphi}d\varphi - S_{eq}d\sigma_1\right) \tag{41}$$

Also,

$$\Delta\varepsilon_m^p = \frac{1}{3\lambda+2\mu}\left(\sigma_m^* - \sigma_2 S_m\right) \Rightarrow d\Delta\varepsilon_m^p = \frac{1}{3\lambda+2\mu}\left(\frac{\partial\sigma_m^*}{\partial\Delta\varepsilon_{kl}}d\Delta\varepsilon_{kl} - \sigma_2\frac{dS_m}{d\varphi}d\varphi - S_m d\sigma_2\right) \tag{42}$$

It goes without saying that the expressions of $\partial\sigma_{eq}^*/\partial\Delta\varepsilon_{kl}$ and $\partial\sigma_m^*/d\Delta\varepsilon_{kJ}$ here are the same as for Gurson model (equations Eq.(16), Eq.(17)). By transferring these expressions into Eq.(40), we obtain:

$$\begin{aligned} d\sigma_1 = &\frac{1}{3\mu}\frac{\partial\sigma_1}{\partial\varepsilon_{eq}}\left(\frac{\partial\sigma_{eq}^*}{\partial\Delta\varepsilon_{kl}}d\Delta\varepsilon_{kl} - \sigma_1\frac{dS_{eq}}{d\varphi}d\varphi - S_{eq}d\sigma_1\right) \\ &+ \frac{\varepsilon}{3\lambda+2\mu}\frac{\partial\sigma_1}{\partial\varepsilon_m}\left(\frac{\partial\sigma_m^*}{\partial\Delta\varepsilon_{kl}}d\Delta\varepsilon_{kl} - \sigma_2\frac{dS_m}{d\varphi}d\varphi - S_m d\sigma_2\right) \end{aligned} \tag{43}$$

By reasoning in the same way for σ_2, we obtain:

$$\begin{aligned} d\sigma_2 = &\frac{1}{3\mu}\frac{\partial\sigma_2}{\partial\varepsilon_{eq}}\left(\frac{\partial\sigma_{eq}^*}{\partial\Delta\varepsilon_{kl}}d\Delta\varepsilon_{kl} - \sigma_1\frac{dS_{eq}}{d\varphi}d\varphi - S_{eq}d\sigma_1\right) \\ &+ \frac{\varepsilon}{3\lambda+2\mu}\frac{\partial\sigma_2}{\partial\varepsilon_m}\left(\frac{\partial\sigma_m^*}{\partial\Delta\varepsilon_{kl}}d\Delta\varepsilon_{kl} - \sigma_2\frac{dS_m}{d\varphi}d\varphi - S_m d\sigma_2\right) \end{aligned} \tag{44}$$

By successively taking $\varphi = cst$ then $\Delta\varepsilon = cst$ in these equations, we obtain the following two systems:

$$\begin{cases} \left(1 + \frac{S_{eq}}{3\mu}\frac{\partial\sigma_1}{\partial\varepsilon_{eq}}\right)\frac{\partial\sigma_1}{\partial\Delta\varepsilon_{kl}} + \frac{\varepsilon S_m}{3\lambda+2\mu}\frac{\partial\sigma_1}{\partial\varepsilon_m}\frac{\partial\sigma_2}{\partial\Delta\varepsilon_{kl}} = \frac{1}{3\mu}\frac{\partial\sigma_1}{\partial\varepsilon_{eq}}\frac{\partial\sigma_{eq}^*}{\partial\Delta\varepsilon_{kl}} + \frac{\varepsilon}{3\lambda+2\mu}\frac{\partial\sigma_1}{\partial\varepsilon_m}\frac{\partial\sigma_m^*}{\partial\Delta\varepsilon_{kl}} \\ \frac{S_{eq}}{3\mu}\frac{\partial\sigma_2}{\partial\varepsilon_{eq}}\frac{\partial\sigma_1}{\partial\Delta\varepsilon_{kl}} + \left(1 + \frac{\varepsilon S_m}{3\lambda+2\mu}\frac{\partial\sigma_2}{\partial\varepsilon_m}\right)\frac{\partial\sigma_2}{\partial\Delta\varepsilon_{kl}} = \frac{1}{3\mu}\frac{\partial\sigma_2}{\partial\varepsilon_{eq}}\frac{\partial\sigma_{eq}^*}{\partial\Delta\varepsilon_{kl}} + \frac{\varepsilon}{3\lambda+2\mu}\frac{\partial\sigma_2}{\partial\varepsilon_m}\frac{\partial\sigma_m^*}{\partial\Delta\varepsilon_{kl}} \end{cases} \tag{45}$$

$$\left\{\begin{array}{l}\left(1+\frac{\mathrm{S_{eq}}}{3\mu}\frac{\partial\sigma_1}{\partial\varepsilon_{\mathrm{eq}}}\right)\frac{\partial\sigma_1}{\partial\varphi}+\frac{\varepsilon S_{\mathrm{m}}}{3\lambda+2\mu}\frac{\partial\sigma_1}{\partial\varepsilon_{\mathrm{m}}}\frac{\partial\sigma_2}{\partial\varphi}=-\frac{\sigma_1}{3\mu}\frac{\partial\sigma_1}{\partial\varepsilon_{\mathrm{eq}}}\frac{\mathrm{dS_{eq}}}{\mathrm{d}\varphi}-\frac{\varepsilon\sigma_2}{3\lambda+2\mu}\frac{\partial\sigma_1}{\partial\varepsilon_{\mathrm{m}}}\frac{\partial\mathrm{S_m}}{\partial\varphi}\\ \frac{\mathrm{S_{eq}}}{3\mu}\frac{\partial\sigma_2}{\partial\varepsilon_{\mathrm{eq}}}\frac{\partial\sigma_1}{\partial\varphi}+\left(1+\frac{\varepsilon\mathrm{S_m}}{3\lambda+2\mu}\frac{\partial\sigma_2}{\partial\varepsilon_{\mathrm{m}}}\right)\frac{\partial\sigma_2}{\partial\varphi}=-\frac{\sigma_1}{3\mu}\frac{\partial\sigma_2}{\partial\varepsilon_{\mathrm{eq}}}\frac{\mathrm{dS_{eq}}}{\mathrm{d}\varphi}-\frac{\varepsilon\sigma_2}{3\lambda+2\mu}\frac{\partial\sigma_2}{\partial\varepsilon_{\mathrm{m}}}\frac{\mathrm{dS_m}}{\partial\varphi}.\end{array}\right. \tag{46}$$

Solving these systems provides the value of the derivatives $\partial\sigma_1/\mathrm{d}\Delta\varepsilon_{\mathrm{kl}}, \partial\sigma_2/\mathrm{d}\Delta\varepsilon_{\mathrm{kl}}, \partial\sigma_1/\partial\varphi, \partial\sigma_2/\partial\varphi$. Note that the matrix 2×2 appearing in the first member is the same for the two systems.

3.2.4. Derivative of φ with respect to $\Delta\varepsilon$

The fundamental equation Eq.(35) giving φ is written:

$$\mathrm{F}(\Delta\varepsilon,\varphi,\sigma_1,\sigma_2)=\mathrm{a}\frac{\sigma_2}{\sigma_1}\left[\sigma^*_{\mathrm{m}}(\Delta\varepsilon)-\sigma_2\mathrm{S_m}(\varphi)\right]\cos\varphi-p\left[\sigma^*_{\mathrm{eq}}(\Delta\varepsilon)-\sigma_1\mathrm{S_{eq}}(\varphi)\right]\mathrm{sh}\left[\frac{3}{2}\mathrm{S_m}(\varphi)\right]=0 \tag{47}$$

Its differentiation gives

$$\frac{\partial\mathrm{F}}{\partial\Delta\varepsilon_{\mathrm{kl}}}\mathrm{d}\Delta\varepsilon_{\mathrm{kl}}+\frac{\partial\mathrm{F}}{\partial\phi}\mathrm{d}\varphi+\frac{\partial\mathrm{F}}{\partial\sigma_1}\mathrm{d}\sigma_1+\frac{\partial\mathrm{F}}{\partial\sigma_2}\mathrm{d}\sigma_2=0 \tag{48}$$

where

$$\frac{\partial\mathrm{F}}{\partial\Delta\varepsilon_{\mathrm{kl}}}=\frac{\partial\mathrm{F}}{\partial\sigma^*_{\mathrm{eq}}}\frac{\partial\sigma^*_{\mathrm{eq}}}{\partial\Delta\varepsilon_{\mathrm{kl}}}+\frac{\partial\mathrm{F}}{\partial\sigma^*_{\mathrm{m}}}\frac{\partial\sigma^*_{\mathrm{m}}}{\partial\Delta\varepsilon_{\mathrm{kl}}},\ \frac{\partial\mathrm{F}}{\partial\sigma^*_{\mathrm{eq}}}=-\mathrm{p}\ \mathrm{sh}\left(\frac{3}{2}S_{\mathrm{m}}\right),\ \frac{\partial\mathrm{F}}{\partial\sigma^*_{\mathrm{m}}}=\mathrm{a}\frac{\sigma_2}{\sigma_1}\cos\varphi, \tag{49}$$

$$\frac{\partial\mathrm{F}}{\partial\varphi}=\mathrm{p}\sigma_1\mathrm{sh}\left(\frac{3}{2}\mathrm{S_m}\right)\frac{\mathrm{dS_{eq}}}{\mathrm{d}\varphi}-\left[\mathrm{a}\frac{\sigma_2^2}{\sigma_1}\cos\varphi+\frac{3}{2}\mathrm{p}\left(\sigma^*_{\mathrm{eq}}-\sigma_{\mathrm{eq}}\right)\mathrm{ch}\left(\frac{3}{2}\mathrm{S_m}\right)\right]\frac{\mathrm{dS_m}}{\mathrm{d}\varphi}-\mathrm{a}\frac{\sigma_2}{\sigma_1}\left(\sigma^*_{\mathrm{m}}-\sigma_{\mathrm{m}}\right)\sin\varphi, \tag{50}$$

$$\frac{\partial\mathrm{F}}{\partial\sigma_1}=-\mathrm{a}\frac{\sigma_2}{\sigma_1^2}\left(\sigma^*_{\mathrm{m}}-\sigma_{\mathrm{m}}\right)\cos\varphi+\mathrm{pS_{eq}}\ \mathrm{sh}\left(\frac{3}{2}S_{\mathrm{m}}\right),\ \frac{\partial\mathrm{F}}{\partial\sigma_2}=\frac{\mathrm{a}}{\sigma_1}\left(\sigma^*_{\mathrm{m}}-2\sigma_{\mathrm{m}}\right)\cos\varphi \tag{51}$$

By expanding $\mathrm{d}\sigma_1=\frac{\partial\sigma_1}{\partial\Delta\varepsilon_{\mathrm{kl}}}\mathrm{d}\Delta\varepsilon_{\mathrm{kl}}+\frac{\partial\sigma_1}{\partial\varphi}\mathrm{d}\varphi$ and $\mathrm{d}\sigma_2=\frac{\partial\sigma_2}{\partial\Delta\varepsilon_{\mathrm{kl}}}\mathrm{d}\Delta\varepsilon_{\mathrm{kl}}+\frac{\partial\sigma_2}{\partial\varphi}\mathrm{d}\varphi$ in Eq.(48), we obtain:

$$\frac{\partial\mathrm{F}}{\partial\Delta\varepsilon_{\mathrm{kl}}}\mathrm{d}\Delta\varepsilon_{\mathrm{kl}}+\frac{\partial\mathrm{F}}{\partial\varphi}\mathrm{d}\varphi+\frac{\partial\mathrm{F}}{\partial\sigma_1}\frac{\partial\sigma_1}{\partial\Delta\varepsilon_{\mathrm{kl}}}\mathrm{d}\Delta\varepsilon_{\mathrm{kl}}+\frac{\partial\mathrm{F}}{\partial\sigma_1}\frac{\partial\sigma_1}{\partial\varphi}\mathrm{d}\varphi+\frac{\partial\mathrm{F}}{\partial\sigma_2}\frac{\partial\sigma_2}{\partial\Delta\varepsilon_{\mathrm{kl}}}\mathrm{d}\Delta\varepsilon_{\mathrm{kl}}+\frac{\partial\mathrm{F}}{\partial\sigma_2}\frac{\partial\sigma_2}{\partial\varphi}\mathrm{d}\varphi=0 \tag{52}$$

and thus

$$\frac{\partial\varphi}{\partial\Delta\varepsilon_{\mathrm{kl}}}=-\frac{\frac{\partial\mathrm{F}}{\partial\Delta\varepsilon_{\mathrm{kl}}}+\frac{\partial\mathrm{F}}{\partial\sigma_1}\frac{\partial\sigma_1}{\partial\Delta\varepsilon_{\mathrm{kl}}}+\frac{\partial\mathrm{F}}{\partial\sigma_2}\frac{\partial\sigma_2}{\partial\Delta\varepsilon_{\mathrm{kl}}}}{\frac{\partial\mathrm{F}}{\partial\varphi}+\frac{\partial\mathrm{F}}{\partial\sigma_1}\frac{\partial\sigma_1}{\partial\varphi}+\frac{\partial\mathrm{F}}{\partial\sigma_2}\frac{\partial\sigma_2}{\partial\varphi}}. \tag{53}$$

Here again, we notice that $\partial\mathrm{F}/\partial\varphi$ is the ”slope” used in Newton’s method to solve Eq.(35).

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