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Representations of Incompressible Vector Fields with Application in Metal Plasticity Modeling

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Contents

1. Introduction

In many concrete problems we are dealing with systems with spherical symmetry. This invariance of the system in the spatial rotations around a given point (most often taken as the origin of the spatial coordinates) makes it possible to write vector fields in a particular form in a coordinate system adapted to this symmetry. In such a coordinate system, the field equations of many models (in domains such electrostatics, electromagnetism, continuum and generalized continuum mechanics etc.) yields simplifications, especially when one tries to solve boundary value problems for the purpose of verifying the accuracy of numerical implementation of these models into a finite element calculations or simply one tries to grasp the equations of the models.

The aim of this report is to study the properties of divergence free vector fields, with particular emphasis on the representation of such vector fields in terms of a "stream" potential function (which may be vector-valued) assuming some cylindrical symmetries. The motivation stems from developing constitutive models to represent ductile failure in porous plastic metals where use is made of a representative volume element in the form of a hollow sphere which is subjected to homogeneous and/or non-homogeneous boundary strain rate, see for instance [4] and references herein. The solution of this problem, due to the axisymmetry of the structure requires the use of the spherical coordinate systems (ρ, θ, ϕ). Due to the symmetries considered the problem becomes independent of the third variable ϕ.

The report presents several representations of the vector fields obtained by taking advantage on the incompressibility property of such fields. These fields are to be used when one tries to find which type of velocity to use that respect the conditions of homogeneous boundary strain rate in a hollow sphere model problem, see for instance [4, 2] and the references herein. Several of such representations were presented and one of them is used to find the gradient of the deformation tensor, which the authors plan to use to in the solution of boundary value problems involving metal plasticity model derived from generalized continuum mechanics approach, Enakoutsa and Leblond [3].

2. Generalities

2.0.1. Boundary conditions of the hollow sphere model problem in the context of Enakoutsa and Leblond [3] micromorphic modeling approach

We shall consider a hollow sphere of centre O, external radius b (representing some kind of average half-spacing between voids) and internal radius $a \equiv bf^{\frac{1}{3}}$, made of some rigid, ideal-plastic von Mises material obeying the normality rule. This sphere is subjected to boundary conditions satisfying the requirements of rotational invariance about the Ox_3 and symmetry to all planes containing that axis. With these conditions the only possible non-zero components of D and ∇D are D_{11}, D_{22} and D_{33} and for ∇D, $D_{11;3} = D_{22;3}$ and $D_{33;3}$. These conditions as well as the consequence they entail are well documented in Enakoutsa and Leblond [3] and are not ducplicated here.

The boundary conditions read

$$\begin{cases} v_1(\mathbf{x}) = D_{11}x_1 = (D_m - D'_{33}/2)x_1 \\ v_2(\mathbf{x}) = D_{11}x_2 = (D_m - D'_{33}/2)x_2 \\ v_3(\mathbf{x}) = D_{11}x_2 = (D_m + D'_{33}/2)x_3 \end{cases} \tag{1}$$

(for $x_1^2 + x_2^2 + x_3^2 = b^2$) where the subscript m and the prime denote the mean and deviatoric part of a second rank-tensor. Besides, using the transformation presented in [3] and [2] the boundary conditions then read

$$\begin{cases} v_1(\mathbf{x}) = D_{11;3}x_1x_3 = (D_{m;3} - D'_{33;3}/2)x_1x_3 \\ v_2(\mathbf{x}) = D_{11;3}x_2x_3 = (D_{m;3} - D'_{33;3}/2)x_2x_3 \\ v_3(\mathbf{x}) = -D_{11;3}(x_1^2 + x_2^2)/2 + D_{33;3}x_3^2/2 \\ = D_{m;3}(x_3^2 - x_1^2 - x_2^2)/2 + D'_{33;3}(x_1^2 + x_2^2 + 2x_2^3)/4 \end{cases} \tag{2}$$

(for $x_1^2 + x_2^2 + x_3^2 = b^2$) where the subscript m and the prime denote the mean and deviatoric part of the third rank-tensor, taken over its symmetric indices. These boundary conditions can be written as in spherical coordinates r, θ, ϕ in the following way

$$\begin{cases} v_r(r = b, \theta, \phi) = \frac{1}{2}D_{m;3}b^2\cos\theta + \frac{1}{4}D'_{33;3}b^2(3\cos^2\theta - 1)\cos\theta \\ v_\theta(r = b, \theta, \phi) = \frac{1}{2}D_{m;3}b^2\sin\theta + \frac{1}{4}D'_{33;3}b^2(3\cos^2\theta + 1)\sin\theta \\ v_\phi(r = b, \theta, \phi) = 0 \end{cases} \tag{3}$$

Incompressible fields satisfying conditions of axisymmetry and symmetry with respect to all vertical planes contains the Ox_3 axis can be represented in the form

$$\mathbf{v}(\mathbf{x}) = \nabla \times [\psi(\mathbf{r}, \theta)\mathbf{e}_\phi] \tag{4}$$

where $\psi(r, \theta)$ represents some "stream" function.

2.0.2. General Representation of Incompressible Vector Fields.

To be more specific, let consider an incompressible vector field $\mathbf{V}$. We have

$$\nabla \cdot \mathbf{V} = 0 \tag{5}$$

and this suggest that

$$\mathbf{V} = \nabla \times \psi \tag{6}$$

where

$$\psi \equiv \psi(r,\theta)\vec{e_\phi} \tag{7}$$

because of the cylindrical symmetry considered. Then, in the basis $(\vec{e_r}, \vec{e_\theta}, \vec{e_\phi})$ we have:

$$\nabla\psi = \begin{pmatrix} 0 & 0 & -\frac{\psi}{r} \\ 0 & 0 & -\frac{\operatorname{cotg}\theta}{r}\psi \\ \psi_{,r} & \frac{1}{r}\psi_{,\theta} & 0 \end{pmatrix} \tag{8}$$

with

$$(\nabla \times \psi)_\alpha = \varepsilon_{\alpha\beta\gamma}(\nabla\psi)_{\gamma\beta}: \tag{9}$$

$$\begin{aligned}
(\nabla \times (\psi))_r &= V_r = \varepsilon_{r\theta\varphi}(\nabla\psi)_{\varphi\theta} + \varepsilon_{r\varphi\theta}(\nabla\psi)_{\theta\varphi} = \frac{1}{2}\psi_{,\theta} + cotg\theta\frac{\psi}{2} \\
(\nabla \times (\psi))_\theta &= V_\theta = \varepsilon_{\theta\varphi r}(\nabla\psi)_{r\varphi} + \varepsilon_{\theta r\varphi}(\nabla\psi)_{\varphi_r} = -\frac{\psi}{r} - \psi_{,r} \\
(\nabla \times (\psi))_\phi &= V_\phi = 0
\end{aligned} \tag{10}$$

Let us assume that

$$\psi(r,\theta) = -\frac{\chi(r,\theta)}{r\sin\theta}; \tag{11}$$

then we have

$$\begin{aligned}
\psi_{,r} &= -\frac{\chi_{,r}}{r\sin\theta} + \frac{\chi}{r^2\sin\theta} \\
\psi_{,\theta} &= -\frac{\chi,\theta}{r\sin\theta} + \frac{\chi\operatorname{cotg}\theta}{r\sin\theta}
\end{aligned} \tag{12}$$

From there we will find the representation of the vector field $\mathbf{V}$ as below:

$$\begin{aligned}
V_r &= -\frac{\chi_{,\theta}}{r^2\sin\theta} + \frac{\chi\operatorname{cotg}\theta}{r^2\sin\theta} - \frac{\chi\operatorname{cotg}\theta}{r^2\sin\theta} = -\frac{\chi_{,\theta}}{r^2\sin\theta} \\
V_\theta &= \frac{\chi}{r^2\sin\theta} + \frac{\chi_{,r}}{r\sin\theta} - \frac{\chi}{r^2\sin\theta} = \frac{x_{,r}}{r\sin\theta}
\end{aligned} \tag{13}$$

Next, let assume that

$$\psi(r,\theta) = \frac{-\chi(r,\theta)}{r\sin\theta}. \tag{14}$$

We want to find a representation of V_r and V_θ as a function of the partial derivative of $\chi(r,\theta)$. Here we have:

$$V_r = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\psi\sin\theta)\right) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{\chi(r,\theta)}{r\sin\theta}\cdot\sin\theta\right) = -\frac{1}{r^2\sin\theta}\frac{\partial\chi(r,\theta)}{\partial\theta}; \tag{15}$$

$$V_\theta = \frac{1}{r}\left(-\frac{\theta}{\partial r}(r\psi)\right) = \frac{1}{r}\left(-\frac{\partial}{\partial r}\left(-\frac{\chi(r,\theta)}{r\sin\theta}\cdot r\right) = \frac{1}{r\sin\theta}\frac{\partial\chi(r,\theta)}{\partial r}\right. \tag{16}$$

With the two previous definition it easy to check that $\nabla\cdot\mathbf{V}=0$. Indeed,

$$\nabla\cdot\mathbf{V} = V_{r,r} + \frac{1}{r}\left(V_{\theta,\theta}+V_r\right) + \frac{1}{r}\left(\text{cotg}\theta V_\theta + V_r\right) \tag{17}$$

and

$$\nabla\cdot\mathbf{V} = -\frac{\chi_{,r\theta}}{r^2\sin\theta} + \frac{2\chi_{,\theta}}{r^3\sin\theta} + \frac{\chi_{,r\theta}}{r^2\sin\theta} - \frac{\chi_{,r}}{r^2\sin\theta}\cos\theta - \frac{2\chi_{,\theta}}{r^3\sin\theta} + \frac{\chi_{,r}}{r^2\sin\theta}\text{cotg}\theta \tag{18}$$

In particular if

$$\chi(r,\theta) = f(r)g(\theta), \tag{19}$$

then we get

$$V_r = -\frac{f(r)g'(\theta)}{r^2\sin\theta}; \tag{20}$$

$$V_\theta = \frac{f'(r)g(\theta)}{r\sin\theta}; \tag{21}$$

In what follows we want to present several examples of velocity fields that can be represented by the

3. Some Examples of Representations of Incompressible Velocity fields

3.1. Preliminary Examples

Let consider the velocity field

$$V = \frac{1}{r^2}\mathbf{e}_r \tag{22}$$

This implies

$$\begin{cases} V_r = \dfrac{1}{r^2} \\ \\ V_\theta = 0 \end{cases} . \tag{23}$$

If we assume that $f(r) = 1$, then Eq.$(100)_2$ is satisfied.
The first one, Eq.$(100)_1$, gives

$$\frac{1}{r^2} = \frac{-g'(\theta)}{r^2 \sin\theta} \tag{24}$$

which implies

$$g'(\theta) = -\sin\theta, \tag{25}$$

yielding

$$g(\theta) = \cos\theta \tag{26}$$

(the integration constant can be omitted here).

In the case where the velocity field satisfies

$$V = -\frac{\rho}{2}\mathbf{e}_\rho + z\mathbf{e}_z \tag{27}$$

we get

$$\begin{cases} V_r = V_\rho \sin\theta + V_z \cos\theta = -\dfrac{\rho}{2}\sin\theta + z\cos\theta = 2\left(\cos^2\theta - \dfrac{\sin^2\theta}{2}\right) = \dfrac{r}{2}(3\cos^2\theta - 1) \\ \\ V_\theta = V_\rho \cos\theta - V_z \sin\theta = -\dfrac{\rho}{2}\cos\theta - z\sin\theta = -\dfrac{3}{2}r\sin\theta\cos\theta \end{cases} \tag{28}$$

Let us assume that $f(r) = r^3$, this gives $f'(r) = 3r^2$. With the second equation Eq. $(28)_2$ we have:

$$-\frac{3}{2}\sin\theta\cos\theta = 3r\frac{g(\theta)}{\sin\theta}. \tag{29}$$

This implies

$$g(\theta) = -\frac{1}{2}\sin^2\theta\cos\theta. \tag{30}$$

Then the first equation Eq. $(28)_1$ is

$$\frac{r}{2}\left(3\cos^2\theta - 1\right) = (-r)\cdot\frac{(-1/2)}{\sin\theta}\left(-\sin^3\theta + 2\sin\theta\cos^2\theta\right) = \frac{r}{2}\left(2\cos^2\theta - \sin^2\theta\right) = \frac{r}{2}\left(3\cos^2\theta - 1\right) \tag{31}$$

which is satisfied.

3.2. Various Representations of the Velocity Fields with $\chi(r,\theta) = f_1(r)g_1(\theta) + f_2(r)g_2(\theta)$

Let us consider the case where

$$\chi(r,\theta) = f_1(r)g_1(\theta) + f_2(r)g_2(\theta) \tag{32}$$

with $f_1(r) = r^3$ and $f_2(r) = r^4$. If we assume $D_{m;3} = 1$ and $D_{33;3} = 0$ in Eq.(3) of the boundary conditions, then we get

$$\begin{cases} \dfrac{b^2}{2}\cos\theta = -b\dfrac{g_1'(\theta)}{\sin\theta} - b^2\dfrac{g_2'(\theta)}{\sin\theta} \\ \dfrac{b^2}{2}\sin\theta = 3b\dfrac{g_1(\theta)}{\sin\theta} + 4b^2\dfrac{g_2(\theta)}{\sin\theta} \end{cases} \tag{33}$$

which implies

$$\begin{cases} \dfrac{1}{2}\sin\theta\cos\theta = -\dfrac{1}{b}g_1'(\theta) - g_2'(\theta) \\ \dfrac{1}{2}\sin^2\theta = \dfrac{3}{b}g_1(\theta) + 4g_2(\theta) \end{cases} \tag{34}$$

Let us now take the derivative of equation Eq.$(34\,)_2$; we get

$$\sin\theta\cos\theta = \frac{3}{b}g_1'(\theta) + 4g_2'(\theta). \tag{35}$$

Combining with Eq.$(34\,)_1$ we get

$$\begin{cases} \dfrac{g_1'(\theta)}{b} + g_2'(\theta) = -\dfrac{1}{2}\sin\theta\cos\theta \\ 3\dfrac{g_1'(\theta)}{b} + 4g_2'(\theta) = \sin\theta\cos\theta \end{cases} \tag{36}$$

Eq.(36) is a system of equations in $g_1'(\theta)$ and $g_1'(\theta)$ which determinant is $D = 1$. Hence,

$$\frac{3}{b}g_1'(\theta) = \begin{vmatrix} -\frac{1}{2}\sin\theta\cos\theta & 1 \\ \sin\theta\cos\theta & 4 \end{vmatrix} = \frac{5}{2}\sin\theta\cos\theta \tag{37}$$

which gives

$$g_1(\theta) = -\frac{3b}{2}\sin^2\theta \tag{38}$$

In addition,

$$g_2'(\theta) = \begin{vmatrix} 1 & -\frac{1}{2}\sin\theta\cos\theta \\ 3 & \sin\theta\cos\theta \end{vmatrix} = \sin\theta\cos\theta + \frac{3}{2}\sin\theta\cos\theta = \frac{5}{2}\sin\theta\cos\theta \tag{39}$$

which implies

$$g_2(\theta) = \frac{5}{4}\sin^2\theta \tag{40}$$

From these two expressions, we observe that g_1 and g_2 are proportional, that is χ is in fact of type $f(r)g(\theta)$. Let find look for solutions of this previous form, f not necessarily being a power function, not even a polynomial we must have

$$\begin{cases} \dfrac{b^2}{2}\cos\theta = -\dfrac{f(b)}{b^2}\dfrac{g'(\theta)}{\sin\theta} \\ \dfrac{b^2}{2}\sin\theta = \dfrac{f'(b)}{b}\dfrac{g(\theta)}{\sin\theta} \end{cases} \tag{41}$$

Eq.(41 $)_2$ yields

$$g(\theta) = \frac{b^2}{2f'(b)}\sin^2\theta \tag{42}$$

which implies that

$$g'(\theta) = \frac{b^3}{f'(b)}\sin\theta\cos\theta. \tag{43}$$

Reporting in Eq.(41 $)_1$ we get

$$\frac{b^2}{2}\cos\theta = -\frac{f(b)}{b^2}\frac{b^3}{f'(b)}\cos\theta \Leftrightarrow \frac{b}{2} = -\frac{f(b)}{f'(b)} \Leftrightarrow f'(b) = -\frac{2}{b}f(b) \tag{44}$$

(The previous solution satisfies this condition: we found $g(\theta) = \sin^2\theta$ and $f(r) = -\frac{3b}{2}r^3 + \frac{5}{4}r^4$
then we have $f(b) = -\frac{b^4}{4}$ and $f'(r) = -\frac{9}{2}br^2 + 5r^3 \Rightarrow f'(b) = \frac{b^3}{2}$ and we indeed have $f'(b) = -\frac{2}{b}f(b)$)
We can also enforce $f(b) = 1$: the solution is an arbitrary f, but with $f(b) = 1$ and $f'(b) = -\frac{2}{b}$ and $g(\theta) = -\frac{b^4}{4}\sin^2\theta$.

3.3. Representation of the Velocity Field with $\chi(r,\theta) = f(r)g(\theta)$

This case corresponds to the more general case where the boundary conditions in Eq.(3) corresponds to the case where $D_{33;3} = 1$ and $D_{m;3} = 0$. We can calculate a velocity field in the form $\chi = f(r)g(\theta)$, with $f(r)$ arbitrary, but not necessarily proportional to r^4. We must then have:

$$\begin{cases} \dfrac{b^2\cos\theta}{4}\left(3\cos^2\theta - 1\right) = -\dfrac{f(b)g'(\theta)}{b^2\sin\theta} \\ \dfrac{-b^2\sin\theta}{4}\left(3\cos^2\theta + 1\right) = \dfrac{f'(\theta)g(\theta)}{b\sin\theta} \end{cases} \tag{45}$$

The equation Eq.(45)$_2$ then yields:

$$g(\theta) = -\frac{b^3 \sin^2\theta}{4f'(b)} \quad \left(3\cos^2\theta + 1\right) \tag{46}$$

which implies that

$$g'(\theta) = -\frac{b^3}{4f'(2)}\left[2\sin\theta\cos\theta\left(3\cos^2\theta + 1\right) \ -6\cos\theta\sin^3\theta\right] \tag{47}$$

$$= -\frac{b^3}{4f'(b)}\left[\sin\theta\cos\theta\left(6\cos^2\theta - 6\sin^2\theta + 2\right)\right] \tag{48}$$

$$= -\frac{b^3}{4f'(b)}\sin\theta\cos\theta\left(-4 + 12\cos^2\theta\right) \tag{49}$$

$$= \frac{b^3}{f'(b)}\sin\theta\cos\theta\left(1 - 3\cos^2\theta\right) \tag{50}$$

$$\tag{51}$$

We then get from Eq.(45)$_1$ that

$$\frac{b^2\cos\theta}{4}\left(3\cos^2\theta - 1\right) = \frac{bf(b)}{f'(b)}\cos\theta\left(3\cos^2\theta - 1\right) \Leftrightarrow \frac{b}{4} = \frac{f(b)}{f'(b)} \Leftrightarrow f'(b) = \frac{4}{b}f(b) \tag{52}$$

(which holds for $f(b) = b^4$), the solution is therefore given, with $f(b) = 1$, by

$$f'(b) = \frac{4}{b} \tag{53}$$

and

$$g(\theta) = -\frac{b^4}{16}\sin^2\theta\left(3\cos^2\theta + 1\right) \tag{54}$$

3.4. Representation of the Velocity Field with $\chi(r,\theta) = f(r)g(\theta), g(\theta) = \sin^2(\theta)h(\cos\theta)$

3.4.1. General Case with $\chi(r,\theta) = f(r)g(\theta), g(\theta) = \sin^2(\theta)h(\cos\theta)$

For this case

$$g'(\theta) = \sin\theta\cos\theta h(\cos\theta) - \sin^3\theta h'(\cos\theta) \tag{55}$$

and this implies that

$$\begin{cases} V_r = \frac{f}{r^2}\left(\sin^2\theta h' - 2\cos\theta h\right) \\ V_\theta = \frac{f'}{2}\sin\theta h \end{cases} \tag{56}$$

Assuming that $f(r) = r^4$ Eq.(56)$_2$ gives

$$r^2\sin\theta\left(\frac{3}{4}\sin^2\theta - 1\right) = 4r^2\frac{g(\theta)}{\sin\theta} \tag{57}$$

from which we have

$$g(\theta) = \frac{\sin^2\theta}{4}\left(\frac{3}{4}\sin^2\theta - 1\right) = -\frac{\sin^2\theta}{16}\left(3\cos^2\theta + 1\right) \tag{58}$$

Thus, after simplification,

$$4g'(\theta) = -\frac{\sin\theta\cos\theta}{2}\left(3\cos^2\theta + 1\right) + \frac{3}{2}\cos\theta\sin^3\theta = \sin\theta\cos\theta\left(1 - 3\cos^2\theta\right) \tag{59}$$

So, the second equation

$$\frac{2^2\cos\theta}{4}\left(3\sin^2\theta - 1\right) = -\frac{r^4}{r^2}\frac{\cos\theta\left(1 - 3\cos^2\theta\right)}{4} \tag{60}$$

is also satisfied.

In addition, with the representation

$$\chi(r,\theta) = f(r)g(\theta), g(\theta) = \sin^2(\theta)h(\cos\theta) \tag{61}$$

we have

$$V_r = \frac{-\frac{\partial\chi}{\partial\theta}(r,\theta)}{r^2\sin\theta} = \frac{-\dfrac{\partial\left(f(r)\sin^2\theta h(\cos\theta)\right)}{\partial\theta}}{r^2\sin\theta} = \frac{f(r)}{r^2}\left(\sin^2\theta h'(\cos\theta) - 2\cos\theta h(\cos\theta)\right) \tag{62}$$

$$V_\theta = \frac{\frac{\partial\chi}{\partial r}(r,\theta)}{r\sin\theta} = \frac{\dfrac{\partial\left(f(r)\sin^2\theta h(\cos\theta)\right)}{\partial r}}{r\sin\theta} = \frac{f'(r)}{r}h(\cos\theta)\sin^2\theta. \tag{63}$$

We can then find the following derivatives

$$V_{\theta,\theta} = \frac{\partial V_\theta}{\partial\theta} = \frac{\partial\left(\dfrac{f'(r)h(\cos\theta)\sin\theta}{r}\right)}{\partial\theta} = \frac{f'(r)}{r}\left(h(\cos\theta)\cos\theta - h'(\cos\theta)\sin h^2\theta\right) \tag{64}$$

$$\begin{aligned} V_{r,\theta} &= \frac{\partial V_r}{\partial\theta} = \frac{\partial\left(\frac{f(r)}{r^2}\left(\sin^2\theta h'(\cos\theta) - 2\cos\theta h(\cos\theta)\right)\right.}{\partial\theta} \\ &= -\frac{f(r)}{r^2}\left(2\sin\theta\cos\theta h'(\cos\theta) - \sin^3\theta h''(\cos\theta) + 2\sin\theta h(\cos\theta) + 2\cos\theta\sin\theta h'(\cos\theta)\right) \end{aligned} \tag{65}$$

$$V_{\theta,r} = \frac{\partial V_\theta}{\partial r} = \frac{\partial\left(\dfrac{f'(r)h(\cos\theta)\sin\theta}{r}\right)}{\partial r} = \frac{f''(r)\cdot r - f'(r)}{r^2}\cdot h(\cos\theta)\sin^2\theta \tag{66}$$

.

and

$$\begin{aligned} V_{r,r} &= \frac{\partial V_r}{\partial r} = \frac{\partial\left(\dfrac{f(r)}{r^2}\left(\sin'\theta h'(\cos\theta) - 2\cos\theta h(\cos\theta)\right)\right.}{\partial r} \\ &= \left(\frac{f'(r)}{r^2} - \frac{2f(r)}{r^3}\right)\left(\sin^2\theta h'(\cos\theta) - 2\cos\theta h(\cos\theta)\right) \end{aligned} \tag{67}$$

As a consequence, the components of the deformation tensor $\mathbf{D}$ are calculated as

$$D_{rr} = V_{r,r} = \left(\frac{f'(r)}{r^2} - \frac{2f(r)}{r^3}\right)\left(\sin^2\theta h'(\cos\theta) - 2\cos\theta h(\cos\theta)\right) \tag{68}$$

$$D_{\theta\theta} = \frac{1}{r}\left(V_{\theta,\theta} + V_r\right) = \left(\frac{f'(r)}{r^2} - \frac{2f(r)}{r^3}\right)h(\cos\theta)\cos\theta + \left(\frac{f(r)}{r^3} - \frac{f'(r)}{r^2}\right)h(\cos\theta)\sin^2\theta \tag{69}$$

$$\begin{aligned} D_{r\theta} &= \frac{1}{2r}\left(V_{r,\theta} - V_\theta\right) + \frac{1}{2}V_{\theta,r} \\ &= \frac{f(r)}{r^3}\left(2h'(\cos\theta)\sin\theta\cos\theta + \sin\theta h(\cos\theta) - \frac{1}{2}\sin^3\theta h''(\cos\theta)\right) \\ &- \frac{f'(r)}{r}h(\cos\theta)\sin^2\theta + \frac{f''(r)}{2r}\sin^2\theta h'(\cos\theta) \end{aligned} \tag{70}$$

$$\begin{aligned} D_{\varphi\varphi} &= \frac{1}{r}\left(V_r + V_\theta \cdot cotg\theta\right) \\ &= \frac{f(r)}{r^3}\left(\sin^2\theta h' c\cos\theta\right) - 2\cos\theta h(\cos\theta)\Big) + \frac{f(r)}{r^2}\sin\theta cotg\theta h(\cos\theta) \\ &= \left(-\frac{2f(r)}{r^3} + \frac{f'(r)}{r^2}\right)h\cos\theta + \frac{f(r)}{r^3}h'(\cos\theta)\sin^2\theta \end{aligned} \tag{71}$$

The calculations give

$$D_{rr} + D_{\theta\theta} + D_{\varphi\varphi} = 0. \tag{72}$$

3.4.2. Particular Case with $\chi(r,\theta) = f(r)g(\theta), g(\theta) = \sin^2(\theta)h(\cos\theta)$ *with* $h\,(cos\theta) = 1$

Now we consider the special case $h\,(cos\theta) = 1$.

If

$$h(\cos\theta) = 1, \tag{73}$$

then

$$\chi(r,\theta) = f(r)\sin^2\theta, \tag{74}$$

and

$$V_r = \frac{-\chi,\theta(r,\theta)}{r^2\sin\theta} = \frac{\frac{\partial\chi}{\partial\theta}(r,\theta)}{r^2\sin\theta}, \quad V_\theta = \frac{\chi,r}{r\sin\theta} = \frac{\frac{\partial\chi}{\partial r}(r,\theta)}{r\sin\theta}. \tag{75}$$

Thus, the components of the velocity field and their derivatives become

$$V_r = \frac{-\frac{\partial\chi}{\partial\theta}(r,\theta)}{r^2\sin\theta} = -\frac{2f(r)\cos\theta}{r^2} \tag{76}$$

$$V_\theta = \frac{\dfrac{\partial\chi}{\partial r}(r,\theta)}{r\sin\theta} = \frac{f'(r)\sin\theta}{r} \tag{77}$$

and

$$V_{r,r} = \frac{\partial V_r}{\partial r} = \frac{2f(r)\cos\theta \cdot 2r - 2f'(r)\cos\theta \cdot r^2}{r^4} \tag{78}$$

$$V_{\theta,\theta} = \frac{\partial v}{\partial \theta} = \frac{f'(r)\cos\theta}{r} \tag{79}$$

$$V_{r,\theta} = \frac{\partial V_r}{\partial \theta} = \frac{2f(r)\sin\theta}{r^2} \tag{80}$$

$$V_{\theta,r} = \frac{\partial V_\theta}{\partial r} = \frac{f''(r)\sin\theta - f'(r)\sin\theta}{r^2}. \tag{81}$$

The components of the deformation tensor are found as

$$D_{rr} = V_{r,r} = \frac{2f(r)\cos\theta \cdot 2r - 2f'(r)\cos\theta \cdot r^2}{r^4} \tag{82}$$

$$D_{\theta\theta} = \frac{1}{r}\left(V_{\theta\theta}, +V_r\right) = \frac{rf'(r)\cos\theta - 2f(r)\cos\theta}{r^3} \tag{83}$$

$$D_{r\theta} = \frac{1}{2r}\left(V_{r,\theta} - V_\theta\right) + \frac{1}{2}V_{\theta,r} = \frac{2f(r)\sin\theta - rf'(r)\sin\theta + f''(r)\sin\theta \cdot r^2 - f'(r)\sin\theta \cdot r}{2r^3} \tag{84}$$

$$D_{\varphi\varphi} = \frac{1}{r}\left(V_r + V_\theta \cot g\theta\right) = \frac{-2f(r)\cos\theta + f'(r)\cos\theta \cdot r}{r^3} \tag{85}$$

Here also, we can check that

$$D_{rr} + D_{\theta\theta} + D_{\varphi\varphi} = 0 \tag{86}$$

confirming the incompressibility property of the velocity field.

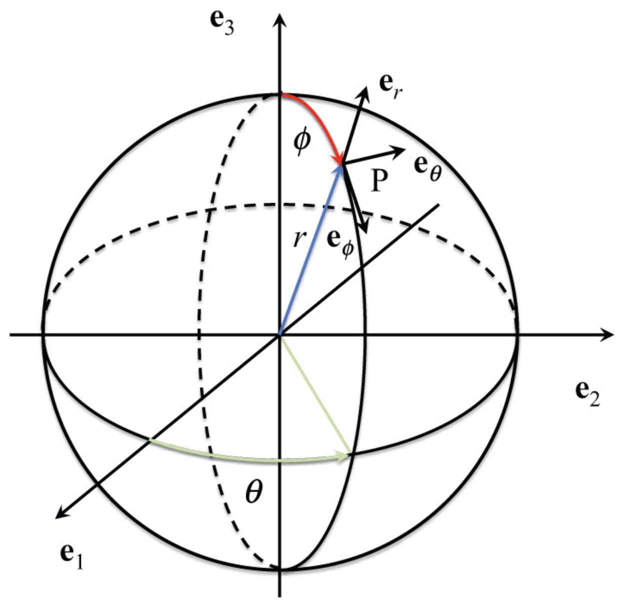

Figure 1: Spherical Coordinates System

4. Gradient of the Deformation Tensor

4.0.1. Generalities

The standard form of the gradient of the deformation is known very well in curvilinear coordinates, in particular in spherical and cylindrical coordinates. We present here such gradient components using covariant differentiation in spherical coordinates system.

The metric tensor for spherical coordinates will be denotes g_{ij}.
We shall consider the spherical coordinates system (r, θ, ϕ) (see Fig. 1) with the mapping

$$\begin{cases} x = r\sin\phi\cos\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\phi \end{cases} \tag{87}$$

The local coordinates system is defined as below:

$$\begin{aligned} \overrightarrow{e_r} &= \frac{\partial}{\partial r}\overrightarrow{OM} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle, \\ \overrightarrow{e_\theta} &= \frac{\partial}{\partial \theta}\overrightarrow{OM} = \langle -r\sin\phi\sin\theta, r\sin\phi\cos\theta, 0\rangle, \\ \overrightarrow{e_\phi} &= \frac{\partial}{\partial \phi}\overrightarrow{OM} = \langle r\cos\phi\cos\theta, r\cos\phi\sin\theta, -r\sin\phi\rangle. \end{aligned} \tag{88}$$

Here

$$g_{ij} = \begin{pmatrix} \overrightarrow{e_r}\cdot\overrightarrow{e_r} & \overrightarrow{e_r}\cdot\overrightarrow{e_\theta} & \overrightarrow{e_r}\cdot\overrightarrow{e_\phi} \\ \overrightarrow{e_\theta}\cdot\overrightarrow{e_r} & \overrightarrow{e_\theta}\cdot\overrightarrow{e_\theta} & \overrightarrow{e_\theta}\cdot\overrightarrow{e_\phi} \\ \overrightarrow{e_\phi}\cdot\overrightarrow{e_r} & \overrightarrow{e_\phi}\cdot\overrightarrow{e_\theta} & \overrightarrow{e_\phi}\cdot\overrightarrow{e_\phi} \end{pmatrix}. \tag{89}$$

Since $\vec{e}_i \cdot \vec{e}_j$ are orthogonal, then we only have $\overrightarrow{e_r} \cdot \overrightarrow{e_r}$, $\overrightarrow{e_\theta} \cdot \overrightarrow{e_\theta}$, $\overrightarrow{e_\phi} \cdot \overrightarrow{e_\phi}$, and the rests are zero.
Therefore,

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2\phi & 0 \\ 0 & 0 & r^2 \end{pmatrix} \tag{90}$$

Next, we determine Christoffel symbols in spherical coordinates.
Using the inverse, g_{ij}^{-1}, of the metric tensor g_{ij},

$$g_{ij}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2 \sin\phi} & 0 \\ 0 & 0 & \frac{1}{r^2} \end{pmatrix}. \tag{91}$$

and the formula for calculating Christoffel symbols,

$$\Gamma_{ij}^m = \frac{1}{2} g^{ml} \left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji} \right), \tag{92}$$

we get

$$\Gamma^r = \begin{pmatrix} \Gamma_{rr}^r & \Gamma_{r\theta}^r & \Gamma_{r\phi}^r \\ \Gamma_{\theta r}^r & \Gamma_{\theta\theta}^r & r_{\theta\phi}^r \\ \Gamma_{\phi r}^r & \Gamma_{\phi\theta}^r & \Gamma_{\phi\phi}^r \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r\sin^2\phi & 0 \\ 0 & 0 & -r \end{pmatrix}; \tag{93}$$

$$\Gamma^\theta = \begin{pmatrix} \Gamma_{rr}^\theta & \Gamma_{r\theta}^\theta & \Gamma_{r\phi}^\theta \\ \Gamma_{\theta r}^\theta & \Gamma_{\theta\theta}^\theta & \Gamma_{\theta\phi}^\theta \\ \Gamma_{\phi r}^\theta & \Gamma_{\phi\theta}^\theta & \Gamma_{\phi\phi}^\theta \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & \cot\phi \\ 0 & \cot\phi & 0 \end{pmatrix}; \tag{94}$$

$$\Gamma^\phi = \begin{pmatrix} \Gamma_{rr}^\phi & \Gamma_{r\theta}^\phi & \Gamma_{r\phi}^\phi \\ \Gamma_{\theta r}^\phi & \Gamma_{\theta\theta}^\phi & \Gamma_{\theta\phi}^\phi \\ \Gamma_{\phi r}^\phi & \Gamma_{\phi\theta}^\phi & \Gamma_{\phi\phi}^\phi \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{r} \\ 0 & -\sin\phi\cos\phi & 0 \\ \frac{1}{r} & 0 & 0 \end{pmatrix}. \tag{95}$$

4.0.2. Gradient of the Deformation Tensor $\mathbf{D}$

Let us determine the gradient of $\mathbf{D}$. From the covariant derivative formula

$$\nabla_l D_{jk} = \partial_l \left(D_{jk} \right) - \Gamma_{jl}^d D_{dk} - \Gamma_{kl}^d D_{jd}, \tag{96}$$

where Γ_{kl}^d represent the Christoffel symbols and $l,\ j, k$ the spherical coordinates r, θ, ϕ, we get

$$\nabla_r D_{jk} = \left\{ \begin{array}{l} \nabla_r D_{rr} = \partial_r D_{rr} \\ \nabla_r D_{r\theta} = \partial_r D_{r\theta} - \frac{1}{r} D_{r\theta} \\ \nabla_r D_{r\phi} = \partial_r D_{r\phi} - \frac{1}{r} D_{r\phi} \\ \nabla_r D_{\theta r} = \partial_r D_{\theta r} - \frac{1}{r} D_{\theta r} \\ \nabla_r D_{\theta\theta} = \partial_r D_{\theta\theta} - \frac{2}{r} D_{\theta\theta} \\ \nabla_r D_{\theta\phi} = \partial_r D_{\theta\phi} - \frac{2}{r} D_{\theta\phi} \\ \nabla_r D_{\phi r} = \partial_r D_{\phi r} - \frac{1}{r} D_{\phi r} \\ \nabla_r D_{\phi\theta} = \partial_r D_{\phi\theta} - \frac{2}{r} D_{\phi\theta} \\ \nabla_r D_{\phi\phi} = \partial_r D_{\phi\phi} - \frac{2}{r} D_{\phi\phi} \end{array} \right\} \tag{97}$$

and

$$\nabla_\theta D_{jk} = \left\{ \begin{array}{l} \nabla_\theta D_{rr} = \partial_\theta D_{rr} - \frac{1}{r} D_{\theta r} - \frac{1}{r} D_{r\theta} \\ \nabla_\theta D_{r\theta} = \partial_\theta D_{r\theta} - \frac{1}{r} D_{\theta\theta} + r \sin^2 \phi D_{rr} + \sin\phi \cos\phi D_{r\phi} \\ \nabla_\theta D_{r\phi} = \partial_\theta D_{r\phi} - \frac{1}{r} D_{\theta\phi} - \cot\phi D_{r\theta} \\ \nabla_\theta D_{\theta r} = \partial_\theta D_{\theta r} + r \sin^2 \phi D_{rr} + \sin\phi \cos\phi D_{\phi r} - \frac{1}{r} D_{\theta\theta} \\ \nabla_\theta D_{\theta\theta} = \partial_\theta D_{\theta\theta} + r \sin^2 \phi D_{r\theta} + \sin\phi \cos\phi D_{\phi\theta} + r \sin^2 \phi D_{\theta r} + \sin\phi \cos\phi D_{\theta\phi} \\ \nabla_\theta D_{\theta\phi} = \partial_\theta D_{\theta\phi} + r \sin^2 \phi D_{r\phi} + \sin\phi \cos\phi D_{\phi\phi} - \cot\phi D_{\theta\theta} \\ \nabla_\theta D_{\phi r} = \partial_\theta D_{\phi r} - \cot\phi D_{\theta r} - \frac{1}{r} D_{\phi\theta} \\ \nabla_\theta D_{\phi\theta} = \partial_\theta D_{\phi\theta} - \cot\phi D_{\theta\theta} + r \sin^2 \phi D_{\phi r} + \sin\phi \cos\phi D_{\phi\phi} \\ \nabla_\theta D_{\phi\phi} = \partial_\theta D_{\phi\phi} - \cot\phi D_{\theta\phi} - \cot\phi D_{\phi\theta} \end{array} \right\} \tag{98}$$

and then

$$\nabla_\phi D_{jk} = \left\{ \begin{array}{l} \nabla_\phi D_{rr} = \partial_\phi D_{rr} - \frac{1}{r} D_{\phi r} - \frac{1}{r} D_{r\phi} \\ \nabla_\phi D_{r\theta} = \partial_\phi D_{r\theta} - \frac{1}{r} D_{\phi\theta} - \cot\phi D_{r\theta} \\ \nabla_\phi D_{r\phi} = \partial_\phi D_{r\phi} - \frac{1}{r} D_{\phi\phi} + r D_{rr} \\ \nabla_\phi D_{\theta r} = \partial_\phi D_{\theta r} - \cot\phi D_{\theta r} - \frac{1}{r} D_{\theta\phi} \\ \nabla_\phi D_{\theta\theta} = \partial_\phi D_{\theta\theta} - 2 \cot\phi D_{\theta\theta} \\ \nabla_\phi D_{\theta\phi} = \partial_\phi D_{\theta\phi} - \cot\phi D_{\theta\phi} + r D_{\theta r} \\ \nabla_\phi D_{\phi r} = \partial_\phi D_{\phi r} + r D_{rr} - \frac{1}{r} D_{\phi\phi} \\ \nabla_\phi D_{\phi\theta} = \partial_\phi D_{\phi\theta} + r D_{r\theta} - \cot\phi D_{\phi\theta} \\ \nabla_\phi D_{\phi\phi} = \partial_\phi D_{\phi\phi} + r D_{r\phi} + r D_{\phi r} \end{array} \right\} \tag{99}$$

after calculations.

Since the non zero components of the velocity field are V_r and V_θ, the deformation tensor and its gradient do not depend on the coordinates ϕ (by symmetry). Thus, the non-zero components of the gradient of the deformation are $\nabla_r D_{rr}$, $\nabla_r D_{r\theta}$, $\nabla_r D_{\theta r}$, $\nabla_r D_{\theta\theta}$, $\nabla_r D_{\phi\phi}$, $\nabla_\theta D_{rr}$, $\nabla_\theta D_{r\theta}$, $\nabla_\theta D_{\theta r}$, $\nabla_\theta D_{\theta\theta}$, $\nabla_\theta D_{\phi\phi}$.

Using the expressions of D_{rr}, $D_{r\theta}$, $D_{\theta\theta}$, and $D_{\phi\phi}$, depending on r and θ, from the previous section, we get the non zero components of the gradient of the deformation as

$$\left\{\begin{array}{l}
\nabla_r D_{rr} = \frac{8f'(r)\cos\theta}{r^3} - \frac{12f(r)\cos\theta}{r^4} - \frac{2f''(r)\cos\theta}{r^2} \\
\nabla_r D_{r\theta} = \frac{4f'(r)\sin\theta}{r^3} - \frac{4f(r)\sin\theta}{r^4} - \frac{2f''(r)\sin\theta}{r^2} + \frac{f'''(r)\sin\theta}{2r} \\
\nabla_r D_{\theta r} = \frac{4f'(r)\sin\theta}{r^3} - \frac{4f(r)\sin\theta}{r^4} - \frac{2f''(r)\sin\theta}{r^2} + \frac{f'''(r)\sin\theta}{2r} \\
\nabla_r D_{\theta\theta} = \frac{10f(r)\cos\theta}{r^4} - \frac{6f'(r)\cos\theta}{r^3} + \frac{f''(r)\cos\theta}{r^2} \\
\nabla_r D_{\phi\phi} = \frac{10f(r)\cos\theta}{r^4} - \frac{6f'(r)\cos\theta}{r^3} + \frac{f''(r)\cos\theta}{r^2} \\
\nabla_\theta D_{rr} = -\frac{4r+2}{r^4} f(r)\sin\theta + \frac{2r+2}{r^3} f'(r)\sin\theta - \frac{f''(r)\sin\theta}{r^2} \\
\nabla_\theta D_{r\theta} = \frac{r+2}{r^4} f(r)\cos\theta - \frac{r+1}{r^3} f'(r)\cos\theta + \frac{f''(r)\cos\theta}{2r} \\
\nabla_\theta D_{\theta r} = \frac{r+2}{r^4} f(r)\cos\theta - \frac{r+1}{r^3} f'(r)\cos\theta + \frac{f''(r)\cos\theta}{2r} \\
\nabla_\theta D_{\theta\theta} = \frac{-f'(r)\sin\theta}{r^2} + \frac{2f(r)\sin\theta}{r^3} \\
\nabla_\theta D_{\phi\phi} = \frac{-f'(r)\sin\theta}{r^2} + \frac{2f(r)\sin\theta}{r^3}
\end{array}\right\} \tag{100}$$

5. Discussion

The representation of the velocity field for the case where $\chi(r,\theta) = f(r)g(\theta), g(\theta) = \sin^2(\theta)h(\cos\theta)$ presented in the sections above can have applications in many domains of interest. As an example it can bring simplification in the calculation of the plastic dissipation while deriving yield limit criterion based on limit-analysis theorem for plastic metals. In such development, use can be made of a representative volume in the form of hollow sphere (see [1], [2], [3]) over which plastic dissipation must be computed and minimised. As an application then we compute the mean value of d_{eq}^2 over the surface of the sphere. We have:

$$\begin{aligned}
d_{eq}^2(r) &\equiv \text{the mean value of } d_{eq}^2 \text{on the surface spherical form} \\
&= \frac{1}{4\pi r^2}\int_0^{\pi} r^2 \sin\theta \int_0^{2\pi} d_{eq}^2 \quad (r,\theta,\varphi) d\varphi \\
&= \frac{1}{2}\int_0^{\pi} d_{eq}^2(r,\theta)\sin\theta d\theta \\
&= \frac{1}{2}\int_{-1}^{1} d_{eq}^2(r,u) du \quad (u=\cos\theta) \\
&\equiv \text{mean value of } d_{eq}^2(u) \text{ in the segment } [-1,1].
\end{aligned} \tag{101}$$

Meanwhile,

$$D^2 rr = \left(\frac{2f}{r^3} - \frac{f'}{r^2}\right)^2 \left(\left(1-u^2\right)^2 (h')^2 + 4u^2h^2 - 4hh'u\left(1-u^2\right)\right) \tag{102}$$

$$D_{\theta\theta}^2 = \left(\frac{2f}{r^3} - \frac{f'}{r^2}\right)^2 u^2h^2 + \left(\frac{f}{r^3} - \frac{f'}{r^2}\right)^2 \left(1-u^2\right)^2 (h')^2 + 2\left(\frac{f'}{r^2} - \frac{2f}{r^3}\right)\left(-\frac{f'}{r^2} + \frac{f}{r^3}\right) hh'u\left(1-u^2\right) \tag{103}$$

$$\begin{aligned}
D_{r\theta}^2 = &\left(\frac{f}{r^3} - \frac{f'}{r^2} + \frac{f''}{2r}\right)^2 \left(1-u^2\right)h^2 + 4\frac{f^2}{r^6}u^2\left(1-u^2\right)(h')^2 + \frac{f^2}{4r^6}\left(1-u^2\right)^3 (h'')^2 \\
&- \frac{f}{r^3}\left(\frac{f}{r^3} - \frac{f'}{r^2} + \frac{f''}{2r}\right)\left(1-u^2\right)hh'' + 4\frac{f}{r^3}\left(\frac{f}{r^3} - \frac{f'}{r^2} + \frac{f''}{2r}\right)hh'u\left(1-u^2\right)^2 - 2\frac{f^2}{r^6}h'h''u\left(1-u^2\right)^2
\end{aligned} \tag{104}$$

$$D_{\varphi\varphi}^2 = \left(\frac{2f}{r^3} - \frac{f'}{r}\right)u^2h^2 + \frac{f^2}{r^6}\left(1-u^2\right)^2 (h')^2 + 2\frac{f}{r^3}\left(-\frac{2f}{r^3} + \frac{f'}{r^2}\right)hh'u\left(1-u^2\right). \tag{105}$$

By setting

$$\begin{aligned}
&A = u^2h^2, \quad B = \left(1-u^2\right)^2 (h')^2, \quad C = u\left(1-u^2\right)hh', \quad D = \left(1-u^2\right)h^2, \\
&E = u^2\left(1-u^2\right)(h')^2, \quad F = \left(1-u^2\right)^3 (h'')^2, \quad G = \left(1-u^2\right)^2 hh'', \quad H = u\left(1-u^2\right)^2 h'h'',
\end{aligned} \tag{106}$$

we get:

$$
\begin{aligned}
d_{eq}^2 &= \frac{4}{3}\left[\left(\frac{2f}{r^3}-\frac{f'}{r^2}\right)^2(B+4A-4C)+\left(\frac{2f}{r^3}-\frac{f'}{r^2}\right)^2 A+\frac{f^2}{r^6}B-2\frac{f}{r^3}\left(\frac{2f}{r^3}-\frac{f'}{r^2}\right)C\right.\\
&+\left(\frac{2f}{r^3}-\frac{f'}{r^2}\right)^2(-2A+C)+\frac{f}{r^3}\left(\frac{2f}{r^3}-\frac{f'}{r^2}\right)(2C-B)+\left(\frac{f}{r^3}-\frac{f'}{r^2}+\frac{f''}{2r}\right)^2 D+4\frac{f^2}{r^6}E+\frac{f^2}{r^6}\frac{F}{4}\\
&\left.-\frac{f}{r^3}\left(\frac{f}{r^3}-\frac{f'}{r^2}+\frac{f''}{2r}\right)G+\frac{f}{r^3}\left(\frac{f}{r^3}-\frac{f'}{r^2}+\frac{f''}{2r}\right)4C-2\frac{f^2}{r^6}H\right]\\
&=\frac{4}{3}\left[(B+4A-4C+A-2A+C)\left(\frac{2f}{r^3}-\frac{f'}{r^2}\right)^2+D\left(\frac{f}{r^3}-\frac{f'}{r^2}+\frac{f''}{2r}\right)^2\right.\\
&+\left(B+4E+\frac{F}{4}-2H\right)\frac{f^2}{r^6}+(-2C+2C-B)\frac{f}{r^3}\left(\frac{2f}{r^3}-\frac{f'}{r^2}\right)\\
&\left.+(-G+4C)\frac{f}{r^3}\left(\frac{f}{r^3}-\frac{f'}{r^2}+\frac{f''}{2r}\right)\right]
\end{aligned}
\tag{107}
$$

which yields:

$$
\begin{aligned}
d_{eq}^2 &= \frac{4}{3}\left[(3A+B-3C)\left(\frac{2f}{r^3}-\frac{f'}{r^2}\right)^2+D\left(\frac{f}{r^3}-\frac{f'}{r^2}+\frac{f''}{2r}\right)^2+\left(B+4E+\frac{F}{4}-2H\right)\frac{f^2}{r^6}\right.\\
&\left.-B\frac{f}{r^3}\left(\frac{2f}{r^3}-\frac{f'}{r^2}\right)+(4C-G)\frac{f}{r^3}\left(\frac{f}{r^3}-\frac{f'}{r^2}+\frac{f'}{2r}\right)\right].
\end{aligned}
\tag{108}
$$

or equivalently

$$
\begin{aligned}
d_{eq}^2 &= \frac{4}{3}\left[\frac{f^2}{r^6}(12A+4B-12C+D+B+4E+\frac{F}{4}-2H-2B+4C-G)+\frac{(f')^2}{r^4}(3A+B-3C+D)\right.\\
&+\frac{f''}{r^2}\cdot\frac{D}{4}+\frac{ff'}{r^5}(-12A-4B+12C-2D+B-4C+G)\\
&\left.+\frac{ff''}{r^4}(D+2C-\frac{G}{2})+\frac{f'f''}{r^3}(-D)\right].
\end{aligned}
\tag{109}
$$

For the special case

$$h\,(cos\theta)=1,$$

we have

$$B=0, E=0, F=0, G=0, H=0.$$

By using Eq.(109) the term d_{eq}^2 becomes

$$
\begin{aligned}
d_{eq}^2 &= \frac{4}{3}\left[\frac{f^2}{r^6}(12A - 12C + D + 4C)\frac{(f')^2}{r^4}(3A - 3C + D) + \frac{f''}{r^2}\cdot\frac{D}{4} + \frac{ff'}{r^5}(-12A + 12C - 2D - 4C)\right.\\
&\left.+\frac{ff''}{r^4}(2C + D) + \frac{f'f''}{r^4}(-D)\right]\\
&= \frac{4}{3}\left[(3A - 3C)\left(\frac{2f}{r^3} - \frac{f'}{r^2}\right)^2 + D\left(\frac{f}{r^3} - \frac{f'}{r^2} + \frac{f''}{2r}\right)^2\right.\\
&\left.+4C\frac{f}{r^3}\left(\frac{f}{r^3} - \frac{f'}{r^2} + \frac{f''}{2r}\right)\right].
\end{aligned}
\tag{110}
$$

The velocity field we found can be used in several applications. For instance, in fluid mechanics, it can be used in the context of Navier-Stokes equations applications to find the pressure distribution around a sphere in an air flow. It can also be used in the context of developing constitutive models for void growth in porous metallic materials. This can be done by using the field we find to obtain a velocity field which must be constant at the boundary of a spherical representative volume and then use it in the limit analysis of a hollow sphere subjected to constant boundary strain rate problem, see for instance Gurson [4] and Enakoutsa [2]

6. Conclusion

We give several representations of a divergence free vector field, with cylindrical symmetries. As an application we used one of the representations to calculate the equivalent deformation over a representative volume element. We also calculate the gradient of the deformation arising from this particular chosen velocity representation. The results found here can be used to derive a constitutive model for damage in porous metals and solving boundary value problems to assess the numerical implementation of the deformation of porous metallic materials models.

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