Bob's contributions to Hopf algebras

In the meddle of 1960's S. Chase and M. Sweedler founded theory of those Hopf algebras which are not necessarily commutative or cocommutative. Nevertheless, the main concern in the early times seemed to investigate cocommutative Hopf algebras, often aiming at applications to characteristic-free study of algebraic groups. There started a movement in the early 1980's when many researchers with their backgrounds in operator algebras, non-commutative rings or representation theory came into the research area, and broadened it. Bob was one of those new comers. Reviewing his contributions (see the references below) we will see two origins of Hopf-Galois Theory.

Working over a field, suppose that a Hopf algebra H acts on an algebra A (so that A is an H-module algebra). Then one constructs the algebra $A\#H$ of smash product, on which every Hopf subalgebra U of the dual Hopf algebra H° acts; indeed, it naturally acts on the factor H in $A\#H$. The main theorem of [1] states that the resulting smash-product algebra $(A\#H)\#U$ is isomorphic $A\otimes(H\#U)$, if H and U satisfy some natural assumptions. Motived by analogous results on operator algebras, this generalizes to a large extent the preceding result by M. Cohen and S. Montgomery when H is a finite group algebra and $U = H^*$. I understand that the basic idea to recover the algebra A from the smash product $(A\#H)\#U$, with its representation-theoretic aspect emphasized, was absorbed into the subsequent theory of Hopf-Galois descent.

In [2] *crossed products* were introduced in the Hopf-algebra context. Let H be a Hopf algebra and B an algebra over a commutative ring. A *crossed product* $B\#_\sigma H$ is the H-comodule algebra which is constructed on the H-comodule $B\otimes H$ by a weak H-action on B and a non-abelian 2-cocycle $\sigma : H\otimes H \to B$. As was defined by Sweedler, an H-comodule algebra A is said to be *cleft* if there exists an H-comodule map $H \to A$ which is invertible with respect to the convolution product. Given an H-comodule algebra A with the subalgebra $B = A^{co\,H}$ of H-coinvariants, Y. Doi and M. Takeuchi [Comm. Algebra **14** (1986), 801–817] proved that (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) hold among the conditions:

(1) A is cleft;
(2) A/B is an H-Galois extension with normal basis property;
(3) A is presented as a crossed product $B\#_\sigma H$.

Generalizing the result proved in [2] in a restricted situation, the article [3] proved (1) $\Leftrightarrow$ (3), so that (1)–(3) are equivalent to each other. We have the dual result on co-cleft H-module algebras and crossed coproducts. In [2] and numerous subsequent papers by other authors it was proved that A is H-cleft or co-cleft for various quotients $A \to H$ or inclusions $H \subset A$ of Hopf algebras or their generalizations. Several of my results are formulated as "this is a crossed (co-)product such that the associated (dual) 2-cocycle vanishes," recent two of which were proved to study super algebraic groups and differential Galois Theory.

Bob was an invaluable person for us who created a happy and friendly atmosphere at many occasions of Hopf-algebra meetings, several of which were co-organized by himself with others. My precious memory, which continues to encourage me, is to have been with Bob and Susan at Mittag-Leffler Institute in Sweden in 2004; it includes my great honor to be invited to the dinner of their wedding anniversary.

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References

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[3] R. J. Blattner, S. Montgomery, *Crossed products and Galois extensions of Hopf algebras*, Pacific J. Math. **137** (1989), 37–54.