Algebra Qualifying Exam

September 2022

You need to complete 8 out of 10 questions
If you write something for more than 8 questions, please indicate
which 8 questions should be graded.

Problem 1. Find all subfields of the field $F = \mathbb{Q}(2^{1/3}, 3^{1/3})$.

Problem 2. Let $P(X) = X^6 + 3$.
   (a) Determine the splitting field of $P(X)$ over $\mathbb{Q}$.
   (b) Determine the isomorphism type of the Galois group of $P(X)$
       over $\mathbb{Q}$.

Problem 3. Let $G$ be a finite group, $p$ a prime number and $H$ a
   subgroup of $G$ with $[G : H] = p$. Assume that no prime number smaller
   than $p$ divides the order of $G$.
   Show that $H$ is normal in $G$.

Problem 4. Let $p$ be a prime number at least 3. Find a set of
   representatives up to conjugation for the group $\text{GL}(2, \mathbb{Z}/p)$ of $2 \times 2$
   invertible matrices.

Problem 5. Let $G$ be the group presented by
   $$G = \langle a, b | a^4 = 1, b^2 = a^2, bab^{-1} = a^{-1} \rangle.$$ 
   You may use that $G$ has order 8. Compute the character table of $G$.

Problem 6. Let $G$ be a finite group, let $V$ be a finite-dimensional
   complex vector space and let $\pi : G \to \text{GL}(V)$ an irreducible represen-
   tation. Let $H$ be an abelian subgroup of $G$.
   Show that $\dim(V) \leq [G : H]$.

Problem 7. Let $S$ be a multiplicatively closed subset of a com-
   mutative ring $R$. Show that for a prime ideal $\mathfrak{p}$ in $R$ disjoint from $S$,
   the ideal $\mathfrak{p} \cdot R[S^{-1}]$ in the localization $R[S^{-1}]$ is prime. Show that this
gives a one-to-one correspondence between prime ideals in $R$ that are
disjoint from $S$ and prime ideals in $R[S^{-1}]$. (A correct proof should
make it clear that you know when two elements of $R[S^{-1}]$ are equal.)
Problem 8. Let $A$ be a commutative ring. Show that the following two statements are equivalent:

(a) every prime ideal of $A$ is equal to an intersection of maximal ideals of $A$

(b) given any ideal $I$ of $A$, the intersection of the prime ideals of $A/I$ is equal to the intersection of the maximal ideals of $A/I$.

Problem 9. Let $\varphi: Ab \to Grp$ be the functor that takes an abelian group $A$ to $A$ in the category of groups. Show that $\varphi$ has a left adjoint $\alpha$. Does $\varphi$ have a right adjoint? Does $\alpha$ have a left adjoint? Justify your answers.

Problem 10. Compute the Jacobson radical $J(R)$ for the following rings $R$. Justify your answers.

(a) Let $R = \text{End}_R(V)$, for a real vector space $V$ of countably infinite dimension. Compute $J(R)$.

(b) For any finite extension field $F$ of $\mathbb{Q}$, let $R$ be the integral closure of $\mathbb{Z}$ in $F$. Compute $J(R)$. 