Algebra Qualifying Exam

September 2022

You need to complete 8 out of 10 questions

If you write something for more than 8 questions, please indicate which 8 questions should be graded.

Problem 1. Find all subfields of the field $F = \mathbb{Q}(2^{1/3}, 3^{1/3})$.

Problem 2. Let $P(X) = X^6 + 3$.

(a) Determine the splitting field of $P(X)$ over $\mathbb{Q}$.

(b) Determine the isomorphism type of the Galois group of $P(X)$ over $\mathbb{Q}$.

Problem 3. Let G be a finite group, p a prime number and H a subgroup of G with $[G : H] = p$. Assume that no prime number smaller than p divides the order of G.

Show that H is normal in G.

Problem 4. Let p be a prime number at least 3. Find a set of representatives up to conjugation for the group $\mathrm{GL}(2, \mathbb{Z}/p)$ of 2×2 invertible matrices.

Problem 5. Let G be the group presented by

$$G = \langle a, b | a^4 = 1, b^2 = a^2, bab^{-1} = a^{-1} \rangle.$$

You may use that G has order 8. Compute the character table of G.

Problem 6. Let G be a finite group, let V be a finite-dimensional complex vector space and let $\pi : G \to \mathrm{GL}(V)$ an irreducible representation. Let H be an abelian subgroup of G.

Show that $\dim(V) \leq [G : H]$.

Problem 7. Let S be a multiplicatively closed subset of a commutative ring R. Show that for a prime ideal $\mathfrak{p}$ in R disjoint from S, the ideal $\mathfrak{p} \cdot R[S^{-1}]$ in the localization $R[S^{-1}]$ is prime. Show that this gives a one-to-one correspondence between prime ideals in R that are disjoint from S and prime ideals in $R[S^{-1}]$. (A correct proof should make it clear that you know when two elements of $R[S^{-1}]$ are equal.)

Problem 8. Let A be a commutative ring. Show that the following two statements are equivalent:

(a) every prime ideal of A is equal to an intersection of maximal ideals of A

(b) given any ideal I of A, the intersection of the prime ideals of A/I is equal to the intersection of the maximal ideals of A/I.

Problem 9. Let $\varphi : \mathcal{A}b \to \mathcal{G}p$ be the functor that takes an abelian group A to A in the category of groups. Show that φ has a left adjoint α. Does φ has a right adjoint? Does α have a left adjoint? Justify your answers.

Problem 10. Compute the Jacobson radical $J(R)$ for the following rings R. Justify your answers.

(a) Let $R = \mathrm{End}_{\mathbb{R}}(V)$, for a real vector space V of countably infinite dimension. Compute $J(R)$.

(b) For any finite extension field F of $\mathbb{Q}$, let R be the integral closure of $\mathbb{Z}$ in F. Compute $J(R)$.