Basic Exam: Fall 2022

Test instructions:

- Write your UCLA ID number on the upper right corner of *each* page.
- Do not write your name anywhere on the exam.
- The final score will be the sum of:

  FIVE linear algebra problems (Problems 1–6) and
  FIVE analysis problems (Problems 7–12).

  However, to pass the exam you need to show mastery of both subjects.
- Indicate below which 10 problems you wish to have graded.
- Please staple your problems in numerical order.

1	2	3	4	5	6
7	8	9	10	11	12

Problem 1. Find and explicit invertible 3×3 matrix U so that

$$UAU^{-1} = A^T \quad \text{in the case that} \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Here A^T denotes the transpose of the matrix A.

Problem 2.(a) Show that the matrix

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

is positive definite.
(b) Determine the volume of the region $\{\vec{x} \in \mathbb{R}^3 : \vec{x} \cdot A\vec{x} \leq 1\}$.
(You may take for granted that the unit sphere has volume $4\pi/3$.)

Problem 3. Let A be an $n \times n$ matrix with real entries that satisfies the equation

$$e^A = I \qquad \text{(here } I \text{ denotes the } n \times n \text{ idenity matrix).}$$

Show, for example via consideration of Jordan Normal Form, that the minimal polynomial of A contains no repeated roots.

Problem 4. Consider the following matrix given in terms of its singular value decomposition:

$$A = \begin{bmatrix} 3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

(a) Rigorously determine the following quantity:

$$M = \inf\left\{\|A\vec{x} - \vec{v}\| : \vec{x} \in \mathbb{R}^4\right\} \quad \text{where} \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Here $\|\cdot\|$ denotes the Euclidean (= Pythagorean) norm.
(b) With M and $\vec{v}$ as above, rigorously determine

$$\ell = \inf\left\{\|\vec{x}\| : \|A\vec{x} - \vec{v}\| = M \text{ and } \vec{x} \in \mathbb{R}^4\right\}$$

Problem 5. Let A be an invertible $n \times n$ matrix with real entries. We wish to consider subspaces W of $\mathbb{R}^n$ with the following property:

$$v^T A w = 0 \quad \text{for all} \quad w, v \in W. \tag{$*$}$$

Here v^T denotes the transpose of the vector v.
(a) Show that any such subspace W satisfies $\dim(W) \leq \frac{n}{2}$.
(b) Show by example that when n is even, there is an invertible matrix A and a subspace W satisfying $(*)$ with $\dim(W) = \frac{n}{2}$.

Problem 6. Let V be a finite-dimensional vector space over a field F, and let V^* denote its dual (i.e., the space of linear maps $f : V \to F$). For a subset S of V, define

$$S^\circ = \{f \in V^* \ : \ f(v) = 0 \text{ for all } v \in S\}$$

Likewise, for a subset X of V^*, define

$$X^\circ = \{v \in V \ : \ f(v) = 0 \text{ for all } f \in X\}$$

Suppose W is a subspace of V.
(a) Prove that $(W^\circ)^\circ = W$ for any subspace W of V.
(b) Prove that $(V/W)^*$ and W° are isomorphic.

Problem 7. Prove that every sequence of real numbers has a monotone (i.e., either non-increasing or non-decreasing) subsequence.

Problem 8. Suppose that for each index $i \in \mathbb{N}$, we are given an $x_i \in [0, 1]$ and a $y_i \in \mathbb{R}$. Suppose further that these pairs satisfy

$$|y_i - y_j| \leq 10|x_i - x_j| \quad \text{for every } i, j \in \mathbb{N}.$$

(a) Show that for any $J \in \mathbb{N}$, there is a function $f_J : [0, 1] \to \mathbb{R}$ so that
  (i) $f_J(x_i) = y_i$ for each $1 \leq i \leq J$, and
  (ii) $|f_J(x) - f_J(x')| \leq 10|x - x'|$ for every $x, x' \in [0, 1]$.

(b) Now show that there is an $f : [0, 1] \to \mathbb{R}$ so that
  (i) $f(x_i) = y_i$ for every $i \in \mathbb{N}$, and
  (ii) $|f(x) - f(x')| \leq 10|x - x'|$ for every $x, x' \in [0, 1]$.

Problem 9. It is natural approximate the circumference of the unit circle by the perimeter of an inscribed regular n-sided polygon:

$$P_n = \sum_{k=1}^{n} \left| e^{2\pi i(k+1)/n} - e^{2\pi i k/n} \right|$$

Rigorously determine the (necessarily unique) values of $\gamma > 0$ and $c \in (0, \infty)$ so that

$$\lim_{n \to \infty} n^\gamma \left[2\pi - P_n \right] = c.$$

Problem 10. Suppose $f : [0,1] \to \mathbb{R}$ satisfies the following:

$$\sum_{i=1}^{n} \left| f(t_i) - f(t_{i-1}) \right|^2 < 100$$

for every choice of $n \in \mathbb{N}$ and of $0 \leq t_0 < t_1 < t_2 < \cdots < t_n \leq 1$. Show that f is Riemann integrable over the interval $[0,1]$.

Problem 11. Let $f : \mathbb{R}^k \to \mathbb{R}^m$ be a a continuous function satisfying

$$|\vec{x}_n| \to \infty \implies |f(\vec{x}_n)| \to \infty.$$

Show that if C is a closed set, then $f(C)$ is also a closed set.

Problem 12. (a) Show that it is possible to uniquely define a sequence x_n of positive numbers so that

$$x_1 = 0 \quad \text{and} \quad x_{n+1}^2 = 9 - x_n.$$

(b) Show that the resulting sequence converges and determine the limit.