QUALIFYING EXAM
Geometry/Topology
Fall 2022

Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. The Grassmannian $\mathrm{Gr}(k,n)$ is the set of all k-dimensional subspaces of $\mathbb{R}^n$. Explicitly construct the structure of a smooth manifold on $\mathrm{Gr}(k,n)$ using atlases. What is its dimension?

2. The orthogonal group $O(n)$ is the set of $n \times n$ matrices M satisfying $M^T M = \mathrm{Id}$. Construct the structure of a smooth manifold on $O(n)$ by viewing it as the preimage of a regular value of a smooth map $\mathbb{R}^{n^2} \to \mathbb{R}^{n(n+1)/2}$. Prove that its tangent bundle is trivializable.

3. Let M be a closed oriented smooth n-manifold. Prove that for every $k \in \mathbb{Z}$, there exists a smooth map $f\colon M \to S^n$ of degree k.

4. Let M be a smooth manifold and let $\omega \in \Omega^1(M)$ be a nowhere vanishing smooth 1-form. Prove that the following are equivalent.

- $\ker(\theta)$ is an integrable distribution.
- $\omega \wedge d\omega = 0$.
- There exists some smooth 1-form $\alpha \in \Omega^1(M)$ satisfying $d\omega = \alpha \wedge \omega$.

5. Let M be a $2n$-dimensional smooth manifold. A symplectic form on M is a smooth closed 2-form $\omega \in \Omega^2(M)$ so that $\omega \wedge \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$ is a volume form (that is, nowhere vanishing). Determine all pairs of positive integers (k,l) so that $S^k \times S^l$ has a symplectic form.

6. Let C_* be a chain complex of free abelian groups. Let $A_* = C_* \otimes \mathbb{Z}/p$ and let $B_* = C_* \otimes \mathbb{Z}/p^2$ be the chain complexes we get by tensoring C_* degreewise with $\mathbb{Z}/p$ and $\mathbb{Z}/p^2$, respectively.

1. Show that we have a short exact sequence of chain complexes

$$0 \to A_* \to B_* \to A_* \to 0$$

induced by the corresponding sequences of abelian groups

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

2. Show how to define a Bockstein natural transformation

$$\beta\colon H_k(A_*) \to H_{k-1}(A_*)$$

such that we have an associated long exact sequence

$$\cdots \to H_k(A_*) \to H_k(B_*) \to H_k(A_*) \xrightarrow{\beta} H_{k-1}(A_*) \to \dots.$$

3. Show that if x and y are elements such that $d(x) = py$, then

$$\beta(\overline{x}) = \bar{y},$$

where the bars indicate the reduction modulo p of the corresponding classes.

4. Show conversely that given an element $\bar{x} \in H_k(A_*)$, if $\beta(\overline{x}) = 0$, then we can find elements $x, y \in C_*$ such that x reduces to $\bar{x}$ modulo p and $d(x) \equiv p^2 y$ modulo p^3.

7. Let H be a union of n lines through the origin in $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 - H)$.

8. Let X be a path connected, locally path connected, semilocally path connected space. Recall that a path connected covering space $\tilde{X} \to X$ is abelian if $\pi_1(\tilde{X})$ is normal in $\pi_1(X)$ and the quotient is abelian. Show that there is an universal abelian cover: this is an abelian cover $\tilde{X} \to X$ such that for any other abelian cover $\tilde{Y} \to X$, there is a covering map $\tilde{X} \to \tilde{Y}$ factoring the map $\tilde{X} \to X$.

9. The space $S^1 \times S^1$ is the mapping cone of the map

$$[a, b] \colon S^1 \to S^1 \vee S^1,$$

representing the commutator of the inclusion of the left summand $a \colon S^1 \to S^1 \vee S^1$ and the inclusion of the right summand $b \colon S^1 \to S^1 \vee S^1$. Use this and the long exact sequence to compute the homology.

10. Let $f \colon X \to Y$ be a continuous, pointed map. Let $\Sigma^n(f) \colon \Sigma^n X \to \Sigma^n Y$ be the nth (pointed) suspension of f. Show that if for some n, $\Sigma^n(f)$ induces the trivial map on reduced homology, then it does for all n.