Qualifying Examination
LOGIC
Fall 2022

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question. For each question, you may use previous questions if needed even if you did not answer them.

Problem 1. A first-order theory T is *small* if $|S_n(\emptyset)| \leq \aleph_0$ for all $n \in \omega$. Show that T is small if and only if for every finite set $A \subseteq \mathcal{M} \models T$, $|S_1(A)| \leq \aleph_0$.

Problem 2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$ and assume that $\mathcal{M}$ is $(|A|+|\mathcal{L}|)^+$-saturated. Show that then $\operatorname{acl}_{\mathcal{M}}(A) = \bigcap \{\mathcal{N} \preceq \mathcal{M} : A \subseteq \mathcal{N}\}$.

Problem 3. Assume that $\mathcal{L}$ is a recursive first-order language and let T be a recursively enumerable $\mathcal{L}$-theory. Show that then T is logically equivalent to a recursive theory T^*.

Problem 4. Let $\mathcal{L}$ be a recursive first-order language and T a consistent recursively enumerable $\mathcal{L}$-theory. Show that T is decidable (i.e. the set $\{\varphi \in \mathcal{L} : T \vdash \phi\}$ is recursive) if and only if there are models $(\mathcal{M}_i : i \in \omega)$ of T such that $T = \{\varphi \in \mathcal{L} : \forall i \in \omega, M_i \vDash \varphi\}$ and an algorithm which determines, for any $i \in \omega$ and any $\mathcal{L}$-sentence φ, whether or not $\mathcal{M}_i \models \varphi$.

Problem 5. Assume $\diamond(\aleph_1)$, and prove that there is an $\aleph_1$-Suslin tree. You may assume that $\diamond(\aleph_1)$ implies the existence of a diamond guessing sequence for subsets of $\omega_1 \times \omega$.

Problem 6. Work in ZF, with*out* the Axiom of Chioce. For a set X, let $\mathcal{G}(X)$ be the collection of all injections of ordinals into X.

(6a) Prove that $\mathcal{G}(X)$ does not inject into X.

(6b) Prove that $\mathcal{G}(X)$ injects into $\mathcal{P}^{(3)}(X)$. $\mathcal{P}^{(3)}$ is the third iteration of the powerset operation.

Problem 7. Classify the following sets in the arithmetical hierarchy:

(7a) $\{e \in \mathbb{N} \mid \varphi_e \text{ is monotone increasing}\}$.

(7b) $\{e \in \mathbb{N} \mid \varphi_e \text{ is monotone increasing except for finitely many exceptions}\}$.

Problem 8. Suppose κ is a measurable cardinal. Let $\mathfrak{A}_\alpha$, $\alpha < \kappa$, be a sequence of models, all in the same countable language, with $\mathfrak{A}_\alpha$ having universe α.

(8a) Let $\pi \colon V \to M$ be elementary with critical point κ. Let $\mathfrak{A}$ be the function $\xi \mapsto \mathfrak{A}_\xi$, and let $\mathfrak{A}^* = \pi(\mathfrak{A})(\kappa)$. Show that there is $\alpha < \kappa$ so that $\mathfrak{A}_\alpha \preceq \mathfrak{A}^*$.

(8b) Prove that there is $\alpha < \beta < \kappa$ so that $\mathfrak{A}_\alpha \preceq \mathfrak{A}_\beta$.