Qualifying Exam, Fall 2022

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) (a) Give a derivation of Newton's method for finding roots of a two times continuously differentiable function $f(x) : \mathrm{R} \to \mathrm{R}$.

(b) A Matlab code that implements Newton's method with initial guess x_0 is applied to some function $f(x)$.

(i) The code returns a result of "-Inf" on the first iteration. What is the most likely explanation for this behaviour and how can it be resolved? Can you give an example of such function f and initial guess x_0?

(ii) The code returns a result of "NaN" on the first iteration. What is the most likely explanation for this behaviour and how can it be resolved? Can you give an example of such function f and initial guess x_0?

[2] (5 Pts.) Let $f(x) : \mathrm{R} \to \mathrm{R}$ be a four times continuously differentiable function with a bound on it's fourth derivative of the form $\sup_{x \in \mathrm{R}} |\frac{\mathrm{d}^4 f}{\mathrm{d}x^4}| \leq \mathrm{M}$. Consider the three point difference approximation to $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ on a uniform mesh with mesh width h of the form

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \approx \mathrm{D}^2(\mathrm{f}, \mathrm{h}, \epsilon) = \frac{(y_{i+1} + \epsilon_{i+1}) - 2\,(y_i + \epsilon_i) + (y_{i-1} + \epsilon_{i-1})}{h^2}$$

for $i = -1, 0, 1$ where $x_i = i\,h$, $y_i = f(x_i)$, and ϵ_i are errors due to the use of finite precision arithmetic.

(a) Derive an error bound for $\left|\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - \mathrm{D}^2(\mathrm{f}, \mathrm{h}, \epsilon)\right|$ assuming that the sign of the errors ϵ_i are such that the error contribution associated with the values of ϵ are maximized.

(b) Determine the relation between h and $|\epsilon|$ that results in a minimal value for the error bound derived in **(a)**.

[3] (5 Pts.) A difference approximation of the first derivative of a function $f(x)$, $\mathrm{D}(h)$, has a local truncation error expansion of the form

$$\mathrm{D}(h) = f'(x) + \mathrm{C}_1 h + O(h^2),$$

where C_1 is a constant. Derive an $O(h^2)$ approximation to $f'(x)$ using Richardson's extrapolation applied to this local truncation error expansion.

[4](a) (5 Pts.) Assume A is a $m \times n$ real valued matrix with $m > n$ and full column rank. Derive the linear system of equations that must be solved to determine the solution $\vec{x} \in \mathrm{R}^n$ that minimizes $||\mathrm{A}\vec{x} - \vec{b}||_2$ for a given $\vec{b} \in \mathrm{R}^m$.

(b) Give a short explanation for the statement "Gaussian elimination (e.g. LU factorization) is, in general, **not** the recommended computational procedure for determining the solution of the equations derived in **(a)**".

[5] (10 Pts.) Consider the second order ODE problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \alpha \frac{\mathrm{d}y}{\mathrm{d}t} + \beta y = 0 \quad (5.1)$$

for $t \in [0, 1]$ with $y(0) = 1$, $\dfrac{\mathrm{d}y}{\mathrm{d}t}(0) = 1$ and $\alpha, \beta \in \mathrm{R}$.

(a) Give the initial value problem associated with the first order system of ODE's that is equivalent to the initial value problem (5.1).

(b) Give a derivation of the leading order terms of the local truncation error expansion that arises when Euler's method is applied to the system of equations in **(a)**.

(c) Derive the region of absolute stability for Euler's method.

(d) For each of the following values of the coefficients α and β determine if the initial value problem in **(a)** is stiff or not-stiff. Explain your answers.

	α	β
Case 1	0	1.0×10^6
Case 2	10	20
Case 3	1.0×10^6	0

(e) When the values of the coefficients α and β result in a stiff system of equations, one considers using an implicit method such as the Trapezoidal method that is A-stable. When using an A-stable method is there any restriction that should be imposed on the timestep? Explain.

[6] **(10 Pts.)** Consider the two linear initial boundary value problems to be solved for $0 < x < \pi$, $0 < t$ with smooth initial data, $u(x,0) = u_0(x)$,

$$\begin{aligned} &(\text{P1}) \qquad u_t = \ \ \cos(x)\, u_x \\ &(\text{P2}) \qquad u_t = -\cos(x)\, u_x \end{aligned}$$

(a) What boundary conditions do you need to impose for each at $x = 0$ and $x = \pi$ to make each a well posed problem?

(b) Set up a stable, convergent finite difference scheme for each of these problems.

Justify your answers.

[7] **(10 Pts.)** Consider the initial boundary value problem to be solved for $0 < x < 1$, $0 < t$ with smooth initial data, $u(x,0) = u_0(x)$,

$$u_t + u^2 u_x = a\, u_{xx} \qquad (a > 0)$$

(a) What boundary conditions do you need to impose at $x = 0$ and $x = 1$ to make this a well posed problem?

(b) Set up a convergent finite difference scheme for this problem that has a time step restriction that diminishes as $a \to 0$.

Justify your answers,

[8] **(10 Pts.)** Consider the problem,

$$\begin{aligned} -\triangle u + u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

in two dimensions, where $f \in L^2(\Omega)$ and Ω is a convex polygonal domain.

(a) Find the weak variational formulation and show that the problem is well-posed by verifying the assumptions of the Lax-Milgram Lemma and by analyzing the appropriate linear and bilinear forms.

(b) Develop and describe the piecewise linear Galerkin finite element approximation to the problem and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution.