Qualifying Exam, Fall 2022

Optimization / Numerical Linear Algebra (ONLA)

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM. PLEASE USE BLANK PAGES AT END FOR ADDITIONAL SPACE.

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1. (10 points) Consider $Ax = b$ with

$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix},$$

where $a \in \mathbb{C}$. Derive the condition on a for Jacobi iterations to converge.

2. (10 points) Consider $Ax = b$ with $A \in \mathbb{R}^{n\times n}$. Assume that $b = e_1 \in \mathbb{R}^n$, and

$$Ae_i = \begin{cases} e_{i+1} & i \le n-1 \\ e_1, & i = n \end{cases},$$

where e_i $(1 \le i \le n)$ are standard basis vectors. Solve $Ax = b$ with GMRES using zero initial guess for x_0. (Describe all intermediate solutions $x_1, x_2, \ldots$.)

3. (10 points) The Conjugate Gradient algorithm can be described as

$$
\begin{aligned}
& r_0 = b - Ax_0, p_0 = r_0, \\
& \text{for } i = 0, 1, 2, \dots \\
& \quad \alpha_i = (r_i^T r_i)/(p_i^T A p_i) \\
& \quad x_{i+1} = x_i + \alpha_i p_i \\
& \quad r_{i+1} = r_i - \alpha_i A p_i \\
& \quad \beta_i = (r_{i+1}^T r_{i+1})/(r_i^T r_i) \\
& \quad p_{i+1} = r_{i+1} + \beta_i p_i
\end{aligned}
$$

Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive **semidefinite** with rank $r < n$ and consider the problem $Ax = b$. Assume $b \neq 0$ and $b \in \text{range}(A)$. Assuming exact arithmetic, prove that CG converges on this problem. (Hint: a possible strategy to approach the proof is to use the diagonalization of A to transform the CG algorithm into an equivalent algorithm that operates in range(A) and null(A) simultaneously).

4. (10 points) Let A be a positive definite symmetric matrix.

(a) Suppose x^* satisfies the system of equations $Ax^* = b$. Prove that x^* is a minimizer of the quadratic form$f(x) = \frac{1}{2}\langle x, Ax\rangle - \langle x, b\rangle$.

(b) Given this fact, explain how the method of gradient descent with exact line search can be used to solve the system $Ax = b$. Write down the steps of the associated method and the conditions under which it converges.

5. (10 points) Consider the (Richardson) iteration:

$$x_{k+1} = \alpha b + (I - \alpha A)x_k,$$

which aims to solve the system $Ax = b$. Prove that if there exists a value α for which this converges, then A must be positive or negative definite.

6. (10 points) Let

$$A = \begin{bmatrix} 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}.$$

(a) If A can be diagonalized via an eigendecomposition $A = X\Lambda X^{-1}$, compute this factorization. If not, explain why it cannot be diagonalized in this way.

(b) Compute a formula for A^x for any integer $x > 0$.

7. (10 points) Consider the following optimization problem in $\mathbb{R}^2$:

$$\begin{aligned} \text{minimize} \quad & \|x - x_0\|^2 \\ \text{subject to} \quad & \|x\|^2 = 4, \end{aligned}$$

where $x_0 = [2, 3]^T$.

(a) Formulate the Lagrange condition for the above problem and find all points satisfying it.

(b) Using second order conditions, determine whether or not each of the candidate points from part (a) are local minimizers.

8. (10 points) **(a)** Suppose $p \in (0,1)$ and consider the function

$$f\colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}; \quad x \mapsto \begin{cases} -\frac{1}{p}x^p, & x \geq 0 \\ \infty, & \text{otherwise.} \end{cases}$$

Show that the Fenchel conjugate of f, defined by $f^*\colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}$; $u \mapsto \sup_{x\in\mathbb{R}} \langle x, u\rangle - f(x)$, is given by

$$f^*\colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}; \quad u \mapsto \begin{cases} -\frac{1}{q}|u|^q, & u < 0 \\ \infty, & \text{otherwise,} \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

(b) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be closed proper convex and f differentiable. Show that a point x^* minimizes $f + g$ if and only if, for any $\lambda > 0$, x^* is a fixed point of the operator

$$(I + \lambda\partial g)^{-1}(I - \lambda\nabla f).$$

Hint. Recall that the subdifferential operator ∂g of a closed proper convex function g is a point-to-set mapping taking x to the subdifferential of g at x, defined by $\partial g(x) = \{y\colon g(z) \geq g(x) + \langle y, (z - x)\rangle \text{ for all } z \in \operatorname{dom} g\}$.