ADE Exam, Spring 2023  
Department of Mathematics, UCLA

1. [10 points]
   (a) Consider the dynamical system
   
   \[
   \begin{align*}
   \frac{dx}{dt} &= y, \\
   \frac{dy}{dt} &= -x - \epsilon x^2 y, \quad (x, y) \in \mathbb{R}^2,
   \end{align*}
   \]
   
   where \( \epsilon \geq 0 \) is a parameter.

   Determine the stability of the equilibrium point at \((0, 0)\).

   (b) Consider the dynamical system
   
   \[
   \begin{align*}
   \frac{dx}{dt} &= y, \\
   \frac{dy}{dt} &= x - x^3 - \delta y + x^2 y, \quad (x, y) \in \mathbb{R}^2,
   \end{align*}
   \]
   
   where \( \delta > 0 \) is a constant.

   Determine the equilibrium points of (2), and use linear stability analysis to classify their type and (when possible) their stability. Show that the two vertical lines \( x = \pm \sqrt{\delta} \) divide the phase plane into three regions such that a periodic orbit cannot exist entirely in one of these regions.

2. [10 points] Consider the Legendre equation
   
   \[
   (1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \ell(\ell + 1)y = 0, \quad -1 \leq x \leq 1,
   \]
   
   where \( \ell \geq 0 \) is an integer. Let \( P_\ell \) denote the solution of (3) that satisfies \( P_\ell(1) = 1 \).

   (a) Show that \( x = 1 \) is a regular singular point. Find the indicial equation and indicial exponents, and find the leading terms of the series expansion at \( x = 1 \) for two linearly independent solutions. Use them to explain why the condition \( P_\ell(1) = 1 \) is sufficient to uniquely determine \( P_\ell \).

   (b) Derive a recursion relation for the coefficients of the series expansion \( y(x) = \sum_{k=0}^{\infty} a_k x^k \) for solutions of (3). Using this relation, show that \( P_\ell \) is a polynomial that (i) consists only of even powers when \( \ell \) is even and (ii) consists only of odd powers when \( \ell \) is odd.
(c) Using the Rodrigues formula

\[ P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} \left( (x^2 - 1)^\ell \right), \]  

(4)

or otherwise, show that \( P_\ell \) satisfies the orthogonality relation

\[ \int_{-1}^{1} P_\ell(x) P_m(x) \, dx = 0, \]  

(5)

and determine the value of the integral (5) when \( \ell = m \).

3. [10 points] Consider the energy functional \( E[u] \), which is defined for \( u \in C^2(D) \) by

\[ E[u] = \frac{1}{2} \int_D (|\nabla u|^2 + u^2) \, d^n x, \]

where \( D \subset \mathbb{R}^n \) is a bounded and open set. Assume that \( u = g(x) \) is known on \( \partial D \).

(a) Derive the partial differential equation that is satisfied by the minimizer of \( E \). Starting from the minimization principle, prove that solutions of this PDE are unique.

(b) Suppose that \( n = 1, D = (-1, 1) \), and \( u(-1) = u(1) = 1 \). Find an approximate solution of your PDE from (a) that takes the form \( u = 1 + A(1 - x^2) \). That is, find the value of \( A \) that minimizes the energy functional.

4. [10 points] Consider the nonlinear partial differential equation

\[ u_t = \Delta u - u^3, \quad x \in D, \quad 0 < t < T, \]

where \( D \subset \mathbb{R}^n \) is a bounded and open set. You may assume that solutions exist and are \( C^{2,1}(D \times (0, T)) \cap C(\overline{D} \times [0, T]) \). Show that the solutions of the PDE are unique.

5. [10 points] Consider the one-dimensional partial differential equation

\[ u_t + \frac{1}{2} u_x^2 = 0, \quad -\infty < x < \infty, \]

with initial condition

\[ u(x, 0) = 0, \quad x < 0; \quad u(x, 0) = 1, \quad x > 0 \]

and boundary conditions

\[ u \to 0 \text{ as } x \to -\infty \quad \text{and} \quad u \to 1 \text{ as } x \to \infty. \]
(a) Show that the PDE does not have a traveling-wave solution that is compatible with these boundary conditions, even when the derivatives are interpreted in the sense of distributions.

(b) Derive the weak solution of the PDE.

[Note: You do not need to derive the Hopf–Lax formula, but you should state the formula carefully if you use it.]

6. [10 points] Consider the partial differential equation

\[ u_t + uu_x = -u, \]

with initial condition

\[ u(x, 0) = 1, \quad x < 0; \quad u(x, 0) = 0, \quad x > 0. \]

(a) Show that for a smooth solution, the PDE can be written in terms of the “characteristic” variable \( x = \xi(t) \) as

\[ \frac{d}{dt} u(\xi(t)) = -u(\xi(t)); \quad \frac{d\xi}{dt} = u. \]

(b) Suppose that \( u(0, 0) \) takes values \( \alpha \in (0, 1) \). Solve for the characteristics \( \xi_\alpha(t) \) starting from \( x = 0 \) with the initial value \( u = \alpha \).

(c) Using the result from (b), solve the Riemann problem with the initial condition above. Write your answer in terms of the Eulerian variables \( x \) and \( t \).

7. [10 points] Consider the Korteweg–de Vries (KdV) equation

\[ u_t + u_{xxx} + 6uu_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad u(x, 0) = f(x). \]

Assume that the function \( u(x, t) \) and all of its derivatives vanish as \( |x| \to \infty \).

(a) Show that the following are conserved quantities in time:

\[ \int_{-\infty}^{\infty} u^2(x, t) \, dx; \quad \int_{-\infty}^{\infty} \left[ \frac{1}{2} u_x^2(x, t) - u^3(x, t) \right] \, dx. \]

(b) Show that the KdV equation does not preserve positivity of the solution by constructing an initial condition \( f \) that is nonnegative for which the solution becomes negative at a later time.

[Hint: Consider a local minimum for which the third derivative in space is nonzero.]
8. [10 points] Solve the initial-value problem

\[ u_{tt} - 2u_{xt} - 15u_{xx} = 0, \]

with \( u(x,0) = g(x) \) and \( u_t(x,0) = h(x) \).

[Hint: Consider factoring the differential operator.]