## ANALYSIS QUALIFYING EXAM, SPRING 2023

## Instructions and rubric

- There are 12 problems: 6 on real analysis, 6 on complex analysis.
- Attempt at most five questions on real analysis and five questions on complex analysis. If you submit answers to more questions than this, please indicate clearly which questions should be graded.
- All questions will be graded out of 10 points. Questions with several parts show the breakdown of points in square brackets.
- In case of partial progress on a problem, details will usually earn more points if they are explained as part of a solution outline for the whole problem.


## REAL ANALYSIS

Problem 1. Let $P$ be the set of all Borel probability measures on $[0,1]$ and let $m$ be Lebesgue measure.
(a) [6 points] Prove that $\mu \in P$ satisfies $\mu \ll m$ if and only if for every $\varepsilon>0$ there exists $\delta>0$ such that
$\int f d \mu<\varepsilon$ whenever $f \in C([0,1]), 0 \leq f \leq 1$ and $\int f d m<\delta$.
[Hint: $\mu+m$ is a measure, so there are results available about approximating one kind of function with another in $L^{1}(\mu+m)$.]
(b) [4 points] Give $P$ the weak-* topology via the Riesz representation theorem identifying $C([0,1])^{*}$, and let

$$
A=\{\mu \in P: \mu \ll m\}
$$

Prove that $A$ is a Borel subset of $P$ for the weak-* topology.
Problem 2. Here are two Banach spaces of real-valued functions on $[0,1]$ :

- Let $C$ be the space of continuous functions with norm

$$
\|f\|_{C}=\sup \{|f(x)|: 0 \leq x \leq 1\}
$$

- Let $L$ be the space of functions $f$ for which the quantity

$$
\|f\|_{L}=|f(0)|+\sup \left\{\frac{|f(x)-f(y)|}{|x-y|}: 0 \leq x<y \leq 1\right\}
$$

is finite. These are called the Lipschitz functions. You may assume without proof that $L$ is a vector space and $\|\cdot\|_{L}$ is a complete norm on $L$, and also that $L \subseteq C$. Let $B_{C}$ and $B_{L}$ be the closed unit balls of $C$ and $L$, respectively.
(a) [3 points] Prove that $B_{L}$ is a closed subset of $C$ for the norm $\|\cdot\|_{C}$.
(b) [5 points] For any $f \in C$, define

$$
\Phi(f)=\int_{0}^{1} f(x)^{4} d x-\left(\int_{0}^{1} f(x) d x\right)^{4}
$$

Prove that $\Phi$ attains its maximum on $B_{L}$.
(c) [2 points] Prove that the functional $\Phi$ from part (b) does not attain its maximum on $B_{C}$. [Hint: We certainly have $\Phi(f) \leq 1$ for $f \in B_{C}$. How close can it get?]

Problem 3. Let $(X, \mathcal{M})$ be a measurable space and let $V$ be a separable real Banach space. (Being 'separable' means that $V$ has a countable dense subset for the norm topology.)
(a) [6 points] Prove that there are dual vectors $L_{1}, L_{2}, \cdots \in V^{*}$ such that $\left\|L_{n}\right\|=1$ for every $n$ and

$$
\|v\|=\sup _{n}\left|L_{n}(v)\right| \quad \text { for all } v \in V \text {. }
$$

[Warning: the separability of $V$ does not imply the separability of $V^{*}$ in general.]
(b) [4 points] Now let $\phi: X \rightarrow V$, and assume that $L \circ \phi$ is measurable from $\mathcal{M}$ to $\mathcal{B}(\mathbb{R})$ for every $L \in X^{*}$. Prove that $\phi$ is measurable from $\mathcal{M}$ to the Borel $\sigma$-algebra generated by the norm topology of $V$. [Hint: start by observing that every open subset of $V$ is a countable union of open balls.]

Problem 4. Let $f_{1}, f_{2}, \ldots$ and $g_{1}, g_{2}, \ldots$ be sequences in the unit ball of $L^{2}([0,1])$, and assume that
(i) $f_{n} \rightarrow f$ and $g_{n} \rightarrow g$ Lebesgue almost everywhere,
(ii) all these functions also lie in the unit ball of $L^{p}([0,1])$ for some $p \in[1, \infty]$.
For which values of $p$ do (i) and (ii) imply that $\left\langle f_{n}, g_{n}\right\rangle \rightarrow\langle f, g\rangle$ ? Justify your answers with proofs or counterexamples.

Problem 5. Let $e_{1}, e_{2}, e_{3}$ be the usual basis of $\mathbb{R}^{3}$, and let $w \in \mathbb{R}^{3}$.
(a) [5 points] Prove that there does not exist any $f \in L^{2}\left(\mathbb{R}^{3}\right)$ such that

$$
f(x)=\frac{1}{6} \sum_{j=1}^{3}\left(f\left(x+e_{j}\right)+f\left(x-e_{j}\right)\right)+e^{i w \cdot x-|x|^{2} / 2} \quad \text { for a.e. } x \text {. }
$$

(b) [5 points] Prove that, for any $\varepsilon>0$, there exists $f \in L^{2}\left(\mathbb{R}^{3}\right)$ such that

$$
\int\left|f(x)-\frac{1}{6} \sum_{j=1}^{3}\left(f\left(x+e_{j}\right)+f\left(x-e_{j}\right)\right)-e^{i w \cdot x-|x|^{2} / 2}\right|^{2} d x<\varepsilon
$$

Problem 6. Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be continuous, let $f \in L^{1}\left(\mathbb{R}^{2}\right)$, and define $A_{r} f(x, y)=\frac{1}{2 r} \int_{-r}^{r} f(x+s, y+\phi(x+s)-\phi(x)) d s \quad\left((x, y) \in \mathbb{R}^{2}, r>0\right)$ whenever the integrand lies in $L^{1}(-r, r)$ as a function of $s$. With this definition, prove that $A_{r} f(x, y)$ exists for Lebesgue almost every $(x, y)$, that it satisfies

$$
\int_{\mathbb{R}^{2}} A_{r} f=\int_{\mathbb{R}^{2}} f \quad \text { for every } r>0
$$

and that $A_{r} f \rightarrow f$ pointwise a.e. as $r \downarrow 0$.
[Hint: start by understanding the case $\phi=0$, and then draw a picture to help you see a reduction of the general case to that one.]

## COMPLEX ANALYSIS

Problem 7. Let $\mathbb{D}=\{z \in \mathbb{C}:|z|<1\}$ and let $f: \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic such that $\sup |f(z)| \leq r$, for some $r<1$.
$|z|<1$
(a) [3 points] Show that $f$ has a fixed point $a \in \mathbb{D}$.
(b) $[7$ points $]$ Let

$$
f^{(n)}=f \circ f \circ \cdots \circ f
$$

be the $n$-fold iterate of $f$. Show that $f^{(n)} \rightarrow a$ uniformly on compact subsets of $\mathbb{D}$.

Problem 8. Let

$$
f(z)=\sum_{n=1}^{\infty} a_{n} z^{n}
$$

be a conformal map from the unit disc $\mathbb{D}=\{z \in \mathbb{C}:|z|<1\}$ onto the domain $\{z \in \mathbb{C}:|\operatorname{Re} z|<1,|\operatorname{Im} z|<1\}$. Show that $a_{n}=0$ for all $n \neq 4 k+1, k=0,1,2 \ldots$

Problem 9. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be entire and set

$$
u(z)=\log \left(1+|f(z)|^{2}\right)
$$

Suppose that

$$
\limsup _{r \rightarrow \infty} \frac{1}{r^{2}} \iint_{|z|<r} u(z) d m(z)<\infty,
$$

where $d m(z)$ denotes the Lebesgue measure on $\mathbb{C}$. Show that $f$ is constant.

Problem 10. Let $\mathbb{D}=\{z \in \mathbb{C}:|z|<1\}$ and let $n \geq 1$ be an integer. A function of the form

$$
B(z)=\lambda \prod_{j=1}^{n} \frac{z-a_{j}}{1-\overline{a_{j}} z}, \quad z \in \mathbb{D}
$$

where $a_{j} \in \mathbb{D}, 1 \leq j \leq n$, and $|\lambda|=1$ is called a Blaschke product of degree $n$. Let $\alpha \in \mathbb{D}$. Show that the function

$$
z \mapsto \frac{B(z)+\alpha}{1+\bar{\alpha} B(z)}
$$

is a Blaschke product of degree $n$.

Problem 11. Show that

$$
\int_{-\infty}^{\infty} \frac{e^{-2 \pi i a t}}{\cosh (\pi t)} d t=\frac{1}{\cosh (\pi a)}
$$

for all $a \in \mathbb{R}$. Here $\cosh (y)=\left(e^{y}+e^{-y}\right) / 2$. Justify all manipulations.
Problem 12. (a) [6 points] Let $f: \mathbb{C} \rightarrow \mathbb{C}$ and $g: \mathbb{C} \rightarrow \mathbb{C}$ be entire functions that satisfy

$$
\begin{equation*}
f^{2}+g^{2}=1, \quad \text { or equivalently, } \quad(f+i g)(f-i g)=1 \tag{1}
\end{equation*}
$$

throughout the complex plane. Show that there exists $h: \mathbb{C} \rightarrow \mathbb{C}$ entire so that

$$
\begin{equation*}
f(z)=\cos (h(z)) \quad \text { and } \quad g(z)=\sin (h(z)) \tag{2}
\end{equation*}
$$

for all $z \in \mathbb{C}$.
(b) [4 points] Let $f: \mathbb{C} \backslash\{0\} \rightarrow \mathbb{C}$ and $g: \mathbb{C} \backslash\{0\} \rightarrow \mathbb{C}$ be holomorphic functions satisfying (1) in $\mathbb{C} \backslash\{0\}$. Show that there need not exist $h: \mathbb{C} \backslash\{0\} \rightarrow \mathbb{C}$ holomorphic such that the representation (2) holds for all $z \in \mathbb{C} \backslash\{0\}$.

