Answer all 10 questions. Each problem is worth 10 points. Justify your answers carefully.

1. Let $M, N$ be smooth manifolds and $F : M \to N$ a smooth proper map.
   
   (a) Show that $F$ maps closed sets to closed sets.
   
   (b) Show that the set of regular values is open.
   
   (c) Let $C \subset N$ be compact. Show that for every open set $U \subset M$ containing $F^{-1}(C)$ there is an open set $V \subset N$ containing $C$, such that $F^{-1}(V) \subset U$.

2. Consider a smooth map $F : \mathbb{CP}^n \to \mathbb{CP}^n$.
   
   (a) When $n$ is even show that $F$ has a fixed point.
   
   (b) When $n$ is odd give an example where $F$ does not have a fixed point.

3. Let
   
   \[
   \omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}
   \]
   
   be a 2-form defined on $\mathbb{R}^3 - \{0\}$ and $S^2 \subset \mathbb{R}^3$ the unit sphere.
   
   (a) Compute $\int_{S^2} i^* \omega$, where $i : S \to \mathbb{R}^3$ is the inclusion.
   
   (b) Compute $\int_{S^2} j^* \omega$, where $j : S^2 \to \mathbb{R}^3$ is defined by $j(x, y, z) = (2x, 3y, 5z)$.

4. Let $M$ be a connected compact manifold with non-empty boundary $\partial M$. Show that $M$ does not retract onto $\partial M$.

5. Let $M^m \subset \mathbb{R}^n$ be a closed connected submanifold of dimension $m$.
   
   (a) Show that $\mathbb{R}^n \setminus M^m$ is connected when $m \leq n - 2$.
   
   (b) When $m = n - 1$ show that $\mathbb{R}^n \setminus M^m$ is disconnected by showing that the mod 2 intersection number $I_2(f, M) = 0$ for all smooth maps $f : S^1 \to \mathbb{R}^n$.

6. 
   
   (a) If $X$ is a finite CW complex and $\tilde{X} \to X$ is a path-connected $n$-fold covering map, then show that the Euler characteristics are related by the formula
   
   \[
   \chi(\tilde{X}) = n\chi(X).
   \]
   
   (b) Let $X = \Sigma_g$ be a closed genus $g$ surface. What path-connected, closed surfaces can cover $X$?
7. A group $G$ is divisible if for all $n$, the map $g \mapsto g^n$ from $G$ to itself is surjective. Show that if $X$ is a path-connected CW-complex and if $\pi_1(X, x)$ is a divisible group, then the only path connected finite cover of $X$ is $X$ itself. (Hint: This can be proven directly or by first showing that a divisible group has no finite index subgroups.)

8. Let $M^n$ be an $n$-manifold, and consider a small disk $D^n$ embedded in $M^n$. Show that the inclusion
\[ \overline{M^n - D^n} \hookrightarrow M^n \]
induces an isomorphism on $\pi_1$ if $n \geq 3$ and a surjection if $n \geq 2$.

9. Find, as a function of $n$ and $m$, the homology groups
\[ H_*(\mathbb{R}P^n \cup \mathbb{R}P^m; \mathbb{Z}). \]

10. Consider the CW-complexes $A = S^n \vee S^n$, $X = S^n \times S^n$, and $B = S^n \times [0, 1]/\ast \times [0, 1]$, where $\ast$ is the basepoint of $S^n$. There are inclusions $A \hookrightarrow X$ given by the pairs of points where at least one is the basepoint and $A \hookrightarrow B$ which takes one $S^n$ to $S^n \times 0$ and the other to $S^n \times 1$. Compute the homology of
\[ Y = X \cup_A B. \]