Qualifying Examination
LOGIC
Spring 2023

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question. For each question, you may use previous questions if needed even if you did not answer them.

Problem 1. Determine which of the following is countably axiomatizable among countable structures. I.e., determine for which of the following classes of structures, there is a countable set of sentences Γ so that a countable structure belongs to the class iff it satisfies Γ.

(1a) Connected graphs.

(1b) Acyclic graphs. Recall that a cycle in a graph is a sequence $x_0, x_1, \dots, x_n$ so that $n \geq 3$, $x_n = x_0$, and for each i, x_i and x_{i+1} are connected by an edge. A graph is acyclic if it has no cycles.

(1c) Graphs whose connected components are complete. (Call a set A of vertices *complete* in a graph G if every two vertices in A are connected by an edge in G.)

(1d) Graphs whose connected components are not complete.

Problem 2. Let T be a theory in a countable languate, and suppose that all the type spaces $S_n(T)$ for $n \in \omega$ are countable. Let $\mathfrak{A} = (A; \dots)$ be a countable model of T. Prove that $\mathfrak{A}$ is saturated iff for every finite $u \subseteq A$ and for every elementary extension $\mathfrak{B}$ of $\mathfrak{A}$, there is an elementary extension $\mathfrak{C}$ of $\mathfrak{B}$ so that $\mathfrak{A}_u$ and $\mathfrak{C}_u$ are isomorphic. (Recall that $\mathfrak{A}_u$ is the expansion of $\mathfrak{A}$ where the elements of u are named by new constants.)

Problem 3. Let φ_n be the nth partial computable function from $\mathbb{N}$ to $\mathbb{N}$. Say that a set $A \subseteq \mathbb{N}$ is an *index set* if for all $n, m \in \mathbb{N}$, if $n \in A$ and $\varphi_n = \varphi_m$, then $m \in A$. Show that $K = \{n \in \mathbb{N} \colon \varphi_n(n) \text{ halts}\}$ is not an index set.

Problem 4. Show that there is no m-complete Δ_2 subset of $\mathbb{N}$. That is, there does not exist an $A \subseteq \mathbb{N}$ such that for all $B \subseteq \mathbb{N}$, B is Δ_2 if and only if $B \leq_m A$. ($B \leq_m A$ stands for B is many-one reducible to A.)

Problem 5. Suppose θ is a sentence that is independent from PA. Show that the set $\{\varphi \colon \varphi \text{ is a sentence and } \mathsf{PA} + \varphi \vdash \theta\}$ is incomputable.

Problem 6. Recall that $\diamond_\kappa$ states that there exists a sequence $\vec{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ with $A_\alpha \subseteq \alpha$, so that every $A \subseteq \kappa$ is "guessed" correctly by $\vec{A}$ stationarily often, meaning that $\{\alpha \mid A \cap \alpha = A_\alpha\}$ is stationary in κ. Prove that $\diamond(\kappa)$ is equivalent to the statement that there exists a sequence $\vec{B} = \langle B_\alpha \mid \alpha < \kappa \rangle$ with $B_\alpha \subseteq \alpha \times \alpha$ so that every $B \subseteq \kappa \times \kappa$ is guessed correctly, meaning that $\{\alpha \mid B \cap \alpha \times \alpha = B_\alpha\}$ is stationary.

Problem 7. Let κ be a regular cardinal of L, and let $H \preceq L_\kappa$. Prove that the transitive collapse of H is a level of L.

Problem 8. Let $\mathcal{L}$ be the language containing a single unary relation symbol. Show that the set of universally valid $\mathcal{L}$-sentences (i.e. $\{\varphi \in \mathcal{L} \colon \vdash \varphi\}$) is computable.