Qualifying Exam, Spring 2023

Optimization / Numerical Linear Algebra (ONLA)

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM. PLEASE USE BLANK PAGES AT END FOR ADDITIONAL SPACE.

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1. (10 points) Consider $Ax = b$ with

$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix},$$

where $a \in \mathbb{C}$. Derive the condition on a for Gauss-Seidel iterations to converge.

2. (10 points) Let $B \in \mathbb{R}^{n \times m}$, $A = I - BB^T$ and $\text{rank}(B) = p$. Assume we are solving $Ax = b$ with Conjugate Gradient, assuming a solution exists, in at most how many iterations would convergence happen?

3. (10 points) Recall that the Lanczos iteration tridiagonalizes a hermitian A by building towards

$$T_n = \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \alpha_3 & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{pmatrix}.$$

The Lanczos algorithm is given by

$$\begin{aligned} &\beta_0 = 0, \quad q_0 = 0, \quad b = \text{arbitrary}, \quad q_1 = b/\|b\| \\ &\text{for } n = 1, 2, 3, \ldots \\ &\qquad v = Aq_n \\ &\qquad a_n = q_n^T v \\ &\qquad v = v - \beta_{n-1} q_{n-1} - \alpha_n q_n \\ &\qquad \beta_n = \|v\| \\ &\qquad q_{n+1} = v/\beta_n \end{aligned}$$

Assume exact arithmetic. Prove that q_j is orthogonal to $q_1, q_2, \ldots, q_{j-1}$.

4. (10 points) Let A be a skew Hermitian matrix (i.e. $A^* = -A$) and let I denote the identity matrix of appropriate size. Prove that $(I - A)^{-1}(I + A)$ is unitary.

5. (10 points) Suppose the matrix $M = AB$ has a QR-factorization. Prove or disprove the following:

   (a) A also has a QR-factorization.

   (b) B also has a QR-factorization.

6. (10 points) Let A be a square diagonalizable matrix with eigenvalues $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots$. Consider the Power Method for computing an eigenvector q_1 corresponding to λ_1, which iterates as $z_{n+1} = Az_n/\|Az_n\|$ (with a randomly selected z_1).

   (a) Show (by example or in words) that if $|\lambda_1| = |\lambda_2|$, the Power Method does not necessarily converge to an eigenvector of λ_1.

   (b) Prove however, that if $\lambda_1 = \lambda_2$ (notice no absolute values here!) and $|\lambda_3| < |\lambda_1|$, then the method still offers convergence to an eigenvector of λ_1.

7. (10 points) Consider the problem to find the extremizers of

$$x_1^2 + x_1 x_2^2 \quad \text{subject to} \quad x_2^3 \leq x_1 \leq 2.$$

Answer the following giving a complete reasoning for the answers:

(a) Write down the KKT conditions for this problem and find all points that satisfy them.

(b) Determine whether or not the points in part (a) satisfy the second order necessary conditions for being local maximizers or minimizers.

(c) Determine whether or not the points in part (b) satisfy the second order sufficient conditions for being local maximizers or minimizers.

8. (10 points)

(a) Consider a linear program in the standard form. Let x be a basic feasible solution. Show that if the reduce cost of every nonbasic variable is positive, then x is the unique solution.

Hint: Recall that a linear program in standard form takes the form minimize $c^T x$ subject to $Ax = b$, $x \geq 0$; a basic feasible solution is an extreme point of the feasible set defined in linear algebraic form; the reduce cost is the unit change in the objective along edge directions.

(b) Use duality to prove the following statement: Consider the problem minimize $c^T x$, $x \in \mathbb{R}^n$, subject to $x \geq 0$. A solution exists if and only if $c \geq 0$. Moreover, if a solution exists, then 0 is a solution.

9. (10 points) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a convex function with $f^* = \inf_x f(x) > -\infty$. Consider the subgradient method

$$x^{(k+1)} = x^{(k)} - \alpha_k g^{(k)}, \quad \text{where} \quad g^{(k)} \in \partial f(x^{(k)}).$$

Show that if $0 < \alpha_k < 2\frac{f(x^{(k)})-f^*}{\|g^{(k)}\|_2^2}$, then

$$\|x^{(k+1)} - x^*\|_2 < \|x^{(k)} - x^*\|_2,$$

for any optimal point x^*.

Hint: Recall that $\partial f(x) = \{g \in \mathbb{R}^n \colon f(y) \geq f(x) + \langle g, y - x\rangle\}$.