Analytical Solutions of Some Benchmark
Problems Incorporating Visco-Elastoplasticity
Laws in Both Small and Large Transformations.

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1 SUMMARY

This note delves into the application of viscous models to address two distinct mechanical behavior problems: firstly, the hydrostatic flow of a hollow sphere utilizing the Norton viscous model, and secondly, the closed-form solution of the toothpaste problem employing an elasto-visco-plastic model.

In the initial scenario, our focus is directed towards comprehending the hydrostatic loading conditions experienced by a hollow sphere. The material matrix of the sphere is presumed to adhere strictly to the Norton visco-plastic rigid behavior model without yield, thereby excluding any elastic effects. The resolution of this problem offers valuable insights for characterizing a homogenized plastic porous material at elevated temperatures, rendering it notably significant for further exploration.

Shifting our attention to the toothpaste problem, our investigation tackles the intricate dynamics of toothpaste flow within a cylindrical tube. The material, exhibiting a consistency between fluidity and solidity, is aptly characterized by a rigid, linear visco-elastic Norton law that incorporates a yield stress. Importantly, the flow is stationary. We meticulously provide all the analytical expressions of the mechanical fields, offering a comprehensive solution to the model. Noteworthy is our discovery that the stress distribution in the tube mirrors exactly that observed in cases where the flow adheres to a Newtonian viscous fluid law. This intriguing correspondence adds a layer of insight into the mechanical behavior of the toothpaste, highlighting the applicability and relevance of the chosen visco-elastic model in capturing essential aspects of the flow dynamics.
The material’s viscous characteristics, particularly evident at elevated temperatures or when subjected to low deformation rates, can be effectively emphasized by employing diverse uni-axial experiments characterized by small deformations. Among these experiments, the simplest involves conducting a tensile test at a specified deformation rate $\dot{\epsilon}$, a procedure easily executed by adjusting the displacement speed of the grips on the tensile machine. It becomes apparent that as the imposed deformation rate $\dot{\epsilon}$ increases, the resulting stress-strain curve $\sigma(\epsilon)$ exhibits a corresponding increase, thereby illustrating the influence of $\dot{\epsilon}$ on the material’s behavior.

At the extreme, when the rate of strain $\dot{\epsilon}$ reaches exceedingly high values, the behavior exhibits quasi-elastic tendencies, leading to the disappearance of visco-plastic deformation due to insufficient time for its occurrence. On the converse end, in the limit as $\dot{\epsilon} \to 0$, two scenarios emerge. In the case of low temperature, the stress-strain curve $\sigma(\epsilon)$ converges towards a non-zero limit, aligning with the characteristic elasto-plastic $\sigma(\epsilon)$ curve. Consequently, no visco-plastic deformation is conceivable beneath this limit curve, denoting the existence of a threshold, which corresponds to the conventional plasticity threshold. Conversely, under high temperatures, no lower limit is discernible for the $\sigma(\epsilon)$ curve as $\dot{\epsilon} \to 0$. As a result, the behavior lacks a threshold, with visco-plastic flow manifesting irrespective of the applied stress, even when it is exceptionally low.

Another enduring experiment revolves around the phenomenon of creep. In this particular trial, a tensile bar undergoes a consistent stress application—an easily attainable condition, achieved, for instance, by merely suspending a weight. In the context of elasto-plastic behavior, the relationship between $\dot{\epsilon}^p$ and $\dot{\sigma}$ and the latter’s nullity affirm the existence of $\dot{\epsilon}^p$ (similarly to $\dot{\epsilon}^e$). Consequently, the total deformation promptly adopts the value $\epsilon = \epsilon^e + \epsilon^p$ and remains constant thereafter. Conversely, in cases of viscous behavior, $\epsilon$ immediately assumes the value $\epsilon^e$ but undergoes subsequent evolution. To elaborate further, the visco-plastic deformation rate $\dot{\epsilon}^{vp}$ experiences an initial decrease (primary creep), followed by an increase (tertiary creep). The ultimate phase corresponds to material damage culminating in final rupture, a facet generally omitted in visco-plastic behavior models, particularly those outlined in the subsequent section.

A modification of the uncomplicated creep test involves the two-stage creep experiment. Commencing with creep under stress $\sigma_1$, one awaits the attainment of secondary creep (at a rate $\dot{\epsilon}^{vp}$), stopping short of reaching tertiary creep. Subsequently, the stress undergoes an abrupt reduction to a value $\sigma_2$. After a certain
duration, a second phase of secondary creep emerges at a strain rate \( \dot{\epsilon}_{vp}^2 < \dot{\epsilon}_{vp}^1 \) (an outcome expected due to the lower magnitude of \( \sigma_2 \) compared to \( \sigma_1 \)). In the transitional stage, if \( \sigma_2 \) is adequately low, the occurrence of creep hesitation might manifest: rather than following a continuous decline from \( \dot{\epsilon}_{vp}^1 \) to \( \dot{\epsilon}_{vp}^2 \) (as might be intuitively anticipated), \( \dot{\epsilon}_{vp}^2 \) abruptly adopts a negative value (even in the presence of \( \sigma_2 \)), then subsequently ascends again until \( \dot{\epsilon}_{vp}^2 > 0 \), passing through the zero point.

In the exploration of uni-axial elasticity, a third experiment delves into the realm of relaxation. This particular experimental scenario involves the instantaneous imposition of a specific deformation, a task easily accomplished. Subsequently, the deformation is held constant, resulting in an immediate stress level dictated by elasticity. Over time, there is a gradual decline in stress, marking the transition from elastic deformation to visco-plastic deformation. The outcome hinges on whether the material exhibits behavior without a threshold, typically observed at high temperatures, or if a threshold is present, typically occurring at low temperatures. In the former scenario, the stress asymptotically approaches zero, while in the latter, it decreases to the threshold value without actually reaching zero.

3 CLOSED-FORM SOLUTION OF THE TOOTHPASTE PROBLEM: ELASTO-VISCO-PLASTIC MODEL FOR MECHANICAL BEHAVIOR

Leveraging the methodologies employed for modeling the elastic-plastic behavior of materials, we integrate the premise of strain rate partitioning, as elucidated in the ensuing equation. This strategic incorporation facilitates a more comprehensive understanding of the material’s mechanical response, accounting for the intricate interplay between strain rates and their respective contributions to the overall deformation process. Drawing upon the methods utilized in modeling the elastic-plastic behavior of materials, we introduce the concept of strain rate partitioning through the subsequent equation:

\[
D = D^e + D^{vp}.
\]  

(1)

This addition enhances the sophistication of our analysis, providing a nuanced framework to discern the nuanced contributions of different strain rates to the overarching deformational dynamics of the material.

In this context, the rate of elastic deformation, denoted as \( D^e \), is intricately governed by a specific hypo-elasticity law. The current predicament lies in formulating an apt expression for the visco-plastic strain, \( D^{vp} \), that not only conforms to the
stipulated conditions but also evolves as a progressively increasing function in response to stress.

Navigating this challenge requires a nuanced understanding of the intricate balance between elastic and visco-plastic behaviors. Crafting an expression for $D^{vp}$ that adheres meticulously to the requirement of a progressively increasing function necessitates a delicate interplay of material properties, demanding a comprehensive synthesis of theoretical foundations and empirical insights.

The simple visco-plastic model we consider here aligns with Norton’s behavior without a yield factor. In the uniaxial scenario, this law presents us with the following relationship:

$$ D^{vp} = \dot{\varepsilon}_0 \left( \frac{\sigma}{\sigma_0} \right)^n $$

In this equation, $\dot{\varepsilon}_0$, $\sigma_0$ and $n$ are treated as constants. It’s important to note that the model, in actuality, is only reliant on two constants, $n$ and $\dot{\varepsilon}_0\sigma_0^{-n}$. We will, however, explain the necessity of introducing a third constant in the subsequent sections for comprehensive understanding.

The exponent $n$ is typically greater than 1, and its value varies significantly, decreasing from high values at lower temperatures to figures closely approximating 1 near the melting point of the material. In many instances, $n$ could be substantially high, on the order of 5 to 10, indicating that materials rarely exhibit linear viscous behavior, especially when contrasted with fluid mechanics.

It’s noteworthy to mention that in this model, if the stress $\sigma$ is kept constant, $D^{vp}$ will also remain unchanged. Consequently, this model is capable of replicating only the second, or stationary, stage of creep, but it fails to accurately simulate primary creep. Furthermore, it doesn’t adequately represent the ‘hesitation’ in creep initiation or the third stage of creep.

To generalize this law to the more complex tri-axial scenario, we shall assume that $D^{vp}$ is colinear with the deviatoric component of the stress, similar to the alignment observed in plasticity theory when the Von Mises yield criterion is applied.
The expression of $D^{vp}$ can be written as follows:

$$D^{vp} = \frac{3}{2} \dot{\varepsilon}_0 \left( \frac{\sigma_{eq}}{\sigma_0} \right)^n \frac{s}{\sigma_{eq}} \sigma_{eq} = \left( \frac{3}{2} s : s \right)^{1/2}$$

This equation demonstrates that $D^{vp}$ is proportional to the deviatoric tensor $s$ and can be utilized to verify the behavior under uniaxial tension load using Eq. (3).

One intriguing property of the Norton visco-plastic model is its reduction to the rigid perfectly plastic Von Mises Model with a yield limit of $\sigma_0$ when we fix $\dot{\varepsilon}_0$ and allow $n$ to approach infinity. This adjustment requires treating $\dot{\varepsilon}_0$ and $\sigma_0$ separately and incorporating them into the single parameter $\dot{\varepsilon}_0 \sigma_0^{-n}$. Consequently, as $n$ tends to infinity, $D^{vp}$ becomes zero for $\sigma_{eq} < \sigma_0$ and infinite for $\sigma_{eq} > \sigma_0$, resulting in unlimited flow when the material yields. As a result, Norton’s model serves as a “visco-plastic regularizer” that mitigates numerical issues such as spurious mesh dependency effects inherent in the Von Mises model.

A more sophisticated variant of the Norton model without yield is the Norton model with yield, which is written as:

$$v^{vp} \equiv \dot{v} = \frac{3}{2} \dot{\varepsilon}_0 \left( \frac{\sigma_{eq}}{\sigma_0} - 1 \right)^n \frac{s}{\sigma_{eq}}$$

where $\langle x \rangle$ denote the “positive part”: $\langle x \rangle = x$ if $x \geq 0$, 0 if $x < 0$. Thus, $v^{vp}$ vanishes if $\sigma_{eq} < \sigma_0$ (existence of a yield) but non-zero whenever $\sigma_{eq} > \sigma_0$ (note that, although a “yield exists” , there is no “criterion” the material should obey, $\sigma_{eq}$ could go beyond the yield $\sigma_0$.). Note that this model depends on 3 parameters $\dot{\varepsilon}_0, \sigma_0, n$. Despite the refinement introduced by the existence of a yield, this model does not reproduce the secondary creep (because $\sigma = cst$ implies that $D^{vp}$ is also constant). It also does not reproduce hesitation in creep. In addition, it reduces to the perfect rigid plastic Von Mises model in the limit $\dot{\varepsilon}_0 \rightarrow \infty$, $\sigma_0$ and $n$ fixed (since $v^{vp}$ vanishes for $\sigma_{eq} < \sigma_0$ and very large for $\sigma_{eq} > \sigma_0$).

There exists other models, more complicated. Among these we shall only mention here the Chaboche model (at least in one of its version), particularly interesting since it is a relatively simple model which can reproduce (contrary to the Norton models) the primary creep phenomenon and the “hesitation” to creep phenomenon. The main equation of the Chaboche model reads:
\[
\begin{align*}
D^{vp} &= \frac{3}{2} \dot{\varepsilon}_0 \left( \frac{\sigma_{eq}}{\sigma_0} \right)^n (s - \alpha) \\
\dot{\alpha} &= \frac{2}{3} h D^{vp} - c\alpha,
\end{align*}
\]

where \( \sigma_{eq} = \left( \frac{3}{2} (s - \alpha) : (s - \alpha) \right)^{1/2} \) is the generalized equivalent stress.

This model introduces an internal (deviatoric) tensorial parameter \( \alpha \), as in plasticity model with accounting for kinematics hardening. It depends on four constants, \( \dot{\varepsilon}_0 \sigma_0^{-n}, n, h, \) and \( c \). The term \(-c\alpha\) in the expression of \( \dot{\alpha} \) (the Jaumann derivative of \( \alpha \)) represents a viscous relaxation of the parameter \( \alpha \), of characteristic time \( \frac{1}{c} \).

To confirm that this model accurately represents both the primary creep phase and the "hesitation" to creep phenomena, we need to formulate the equations for the uni-axial case, considering the assumption of small deformations.

Given the configuration of \( D^{vp} \equiv \dot{\varepsilon}^{vp}, \sigma, \) and \( s \) in such a scenario (as outlined previously), and taking into account that \( \alpha = \alpha e_1 \otimes e_1 - \frac{\alpha}{2} e_2 \otimes e_2 - \frac{\alpha}{2} e_3 \otimes e_3 \) and \( \sigma_{eq} = ||(\sigma - \frac{3}{2} \alpha)|| \), we can reformulate the equations as follows:

\[
\begin{align*}
\dot{\varepsilon}^{vp} &= \dot{\varepsilon}_0 \left( \frac{||\sigma - \frac{3}{2} \alpha||}{\sigma_0} \right)^n \frac{\sigma - \frac{3}{2} \alpha}{||\sigma - \frac{3}{2} \alpha||} = \dot{\varepsilon}_0 \left( \frac{||\sigma - \frac{3}{2} \alpha||}{\sigma_0} \right)^n \text{sgn} (\sigma - \frac{3}{2} \alpha) \\
\dot{\alpha} &= \frac{2}{3} h \dot{\varepsilon}^{vp} - c\alpha,
\end{align*}
\]

Initially (\( \alpha = 0 \)), and therefore, we have \( \dot{\varepsilon}^{vp} = \dot{\varepsilon}_0^{vp} = \dot{\varepsilon}_0 \left( \sigma / \sigma_0 \right)^n \). But later on \( \alpha \) increases, and thus \( \dot{\varepsilon}^{vp} \) decreases (primary creep). The secondary creep is reached when \( \alpha \) a stationary value \( \alpha_{\infty} \) (\( \dot{\alpha} \)), \( \dot{\varepsilon}^{vp} = \dot{\varepsilon}_{\infty}^{vp} \) become constant, less that \( \dot{\varepsilon}_0^{vp} \).

To analyze the "hesitation" to creep phenomena, let’s consider that we’ve arrived at the secondary creep stage, corresponding to the stress \( \sigma_1 \). In this case, \( \alpha \) and \( \dot{\varepsilon}^{vp} \) are constant and equivalent to \( \alpha_{\infty} \) and \( \dot{\varepsilon}_{\infty}^{vp} \), respectively. Now, assume that a secondary stress, denoted by \( \sigma_2 \) and less than \( \frac{3}{2} \alpha_{\infty} \), is applied. Under such circumstances, \( \dot{\varepsilon}_{\infty}^{vp} \) turns negative, illustrating the hesitation or resistance to the creep phenomenon. However, as the time passes by \( \alpha \) goes toward a second stationary value \( \alpha_{2,\infty} \), and \( \dot{\varepsilon}^{vp} \) goes toward \( \dot{\varepsilon}_{2,\infty}^{vp} \), which is positive but less than \( \dot{\varepsilon}_{1,\infty}^{vp} \).
To delve into more details, we aim to derive the precise equations representing primary and secondary creep stages, specifically in scenarios where $n = 1$. Not only does this value align well with physical reality, but it also facilitates analytical computation. In such cases, the term $(\sigma - \frac{3}{2}\alpha)$ consistently stays positive, thereby allowing us to express the flow rule as follows:

$$\varepsilon^{vp} = \frac{\dot{\varepsilon}_0}{\sigma_0} \left( \sigma - \frac{3}{2}\alpha \right)$$  \hspace{1cm} (7)

Using the expression of $\dot{\varepsilon}^{vp}$ in the evolution equation of $\alpha$, we get

$$\dot{\alpha} = \frac{2}{3} \frac{\dot{\varepsilon}_0}{\sigma_0} \left( \sigma - \frac{3}{2}\alpha \right) - c\alpha = \frac{2}{3} \frac{\dot{\varepsilon}_0}{\sigma_0} - \left( c + \frac{\dot{\varepsilon}_0}{\sigma_0} \right) \alpha$$  \hspace{1cm} (8)

The stationary solution of this equation can be written as:

$$\alpha = \alpha_{\infty} \equiv \frac{2}{3} \frac{\dot{\varepsilon}_0}{\sigma_0} \frac{\sigma}{\sigma_0 + c}$$  \hspace{1cm} (9)

Using the initial condition $\alpha(0) = 0$, the intermediate solution can be written as

$$\alpha = \alpha_{\infty} \left( 1 - e^{-\left( \frac{\dot{\varepsilon}_0}{\sigma_0 + c} \right) t} \right) = \frac{2}{3} \frac{\dot{\varepsilon}_0}{\sigma_0} \frac{\sigma}{\sigma_0 + c} \left( 1 - e^{-\left( \frac{\dot{\varepsilon}_0}{\sigma_0} \right) \sigma \left( \frac{\dot{\varepsilon}_0}{\sigma_0 + c} \right) t} \right)$$  \hspace{1cm} (10)

The value of $\dot{\varepsilon}^{vp}$ is obtained using the flow rule:

$$\dot{\varepsilon}^{vp} = \frac{\dot{\varepsilon}_0}{\sigma_0} \left( \frac{\dot{\varepsilon}_0}{\sigma_0} \right) e^{-\left( \frac{\dot{\varepsilon}_0}{\sigma_0 + c} \right) t} + c$$  \hspace{1cm} (11)
From the above discussion, we observe that \( \dot{\varepsilon}_{vp} \) varies with time, transitioning from an initial value of \( \dot{\varepsilon}_{vp}^{0} = \dot{\varepsilon}_{0} \sigma_{0} \sigma_{0} \) to a final value of \( \dot{\varepsilon}_{vp}^{\infty} = \dot{\varepsilon}_{0} \sigma_{0} \sigma_{0} + c \), which is less than \( \dot{\varepsilon}_{vp}^{0} \).

This analysis highlights the role of the viscous relaxation of \( \alpha \) in producing the secondary creep phenomena. In the absence of this factor (i.e. \( c = 0 \)), the equation would result in \( \dot{\varepsilon}_{vp}^{\infty} = 0 \), suggesting that the model could only account for the primary creep stage, and not the secondary one.

4 APPLICATION OF THE NORTON VISCOS MODEL: PROBLEM OF THE FLOW OF TOOTHPASTE WITHIN A CYLINDRICAL TUBE.

We are investigating the flow of toothpaste within a cylindrical tube, aligned along the \( Oz \) axis, with a radius \( R \), see Figure 1. This material, as its name suggests, possesses a pasty consistency, meaning it falls between a fluid and a solid. It’s adequately described by a rigid, linear visco-elastic Norton law (with \( n = 1 \)) that includes yield stress. The flow under consideration is of a stationary nature, and the only non-zero component of the velocity is \( v_z = v(x, z) \). This component is zero on the inner wall of the tube, mirroring the behavior of a viscous fluid. We assume that \( v > 0 \), signifying the fluid flows to the right.

In the context of incompressible materials, where elasticity is neglected and the incompressible viscoelasticity rule is applied, the dynamics are governed by certain constraints. One crucial implication of these constraints is the incompressibility of the material, which has significant consequences on the behavior of the velocity variable \( v \).

In the scenario described, the incompressibility condition imposes restrictions on how the velocity variable \( v \) can vary. Specifically, due to the incompressibility assumption, the velocity variable \( v \) becomes solely dependent on the radial coordinate \( r \). This restriction is essential in understanding the material’s response to deformation, as it simplifies the representation of velocity in the system.

Digging deeper into the mechanics, when considering the strain tensor—a fundamental quantity characterizing deformation—the incompressibility assumption further narrows down the non-zero components. In this case, the only non-zero
component of the strain tensor \( d_{rz} = \frac{1}{2} \frac{dv}{dr} \). This expression signifies the rate of change of velocity with respect to the radial coordinate, emphasizing the axial-radial coupling inherent in the system.

By recognizing and analyzing these constraints imposed by incompressibility, researchers and engineers can gain valuable insights into the material’s behavior and design more accurate models for predicting its response under various conditions. The simplifications introduced by neglecting elasticity and applying incompressible viscoelasticity rules pave the way for a focused understanding of the mechanics at play, facilitating the development of effective solutions and optimizations in diverse engineering and scientific applications.

First, let us consider the visco-plastic zone (where \( \sigma_{eq} > \sigma_0 \)). In this zone, \( \left( \frac{\sigma_{eq}}{\sigma_0} - 1 \right) \neq 0 \) and therefore the flow rule implied that the only non-zero component of \( \mathbf{v} \) is \( s_{rz} \equiv \sigma_{rz} < 0 \) (since it is clear that from the physics point of view \( d_{zr} = \frac{1}{2} \frac{dv}{dr} < 0 \), see Figure 1). Thus, \( \sigma_{eq} = \left( \frac{3}{2} \sigma_{rz}^2 \right)^{1/2} = -\sqrt{3} \sigma_{rz} \) so that the
component “rz” of the flow rule can be written as
\[ d_{rz} = \frac{1}{2} \frac{dv}{dr} = \frac{\sqrt{3}}{2} \frac{\dot{\varepsilon}_0}{\sigma_0} \left( \frac{\sqrt{3} \sigma_{rz}}{\sigma_0} + 1 \right), \] (12)
after substitution and simplification. From Eq.(12) we deduce that
\[ \sigma_{rz} = \sigma_0 \left( \frac{1}{3} \frac{dv}{\dot{\varepsilon}_0 \, dr} - \frac{1}{\sqrt{3}} \right), \] (13)
which shows that \( \sigma_{rz} \) only depends on the coordinate \( r \).

Next, let’s find the pressure \( p = -\frac{1}{3} \text{tr}(\sigma) \) and the velocity fields in the visco-plastic zone. Due to the cylindrical properties involved in the problem, the pressure \( p \) only depends on \( r \) and \( z \). The radial and axial balance equations (note that the orhotradial equation is automatically satisfied) are as follows:
\[
\begin{align*}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta}}{r} &= 0 \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rz}}{r} &= 0,
\end{align*}
\] (14)
which implies
\[
\begin{align*}
\frac{\partial p}{\partial r} &= 0 \\
\frac{\partial \sigma_{rz}}{\partial r} - \frac{dp}{dz} + \frac{\sigma_{rz}(r)}{r} &= 0
\end{align*}
\] (15)
given that \( \sigma = -p I + s \). Thus, the pressure depends only on the coordinate \( z \). In addition, the second equation of Eq. (25) is written as
\[
\left( \frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}(r)}{r} \right) (r) = \frac{dp}{dz} (z),
\] (16)
which implies that the left and right hand sides of these equations equal the same constant, say \( -P \) which represent the drop of the pressure (due to the viscosity) by unit length of the tube. Clearly, \( P > 0 \) for a flow towards the right. Given the expression of \( \sigma_{rz} \) as a function of \( v \), we get:
\[
\sigma_0 \left( \frac{1}{3} \frac{d^2 v}{\dot{\varepsilon}_0 \, dr^2} + \frac{1}{3} \frac{dv}{\dot{\varepsilon}_0 \, dr} - \frac{1}{\sqrt{3}r} \right) = -P
\] (17)
which implies that
\[
\frac{d^2v}{dr^2} + \frac{1}{r} \frac{dv}{dr} = -3 \dot{\varepsilon}_0 \frac{P}{\sigma_0} + \sqrt{3} \frac{\dot{\varepsilon}_0}{r}.
\] (18)

Let’s find a quadratic solution in \(r\) (by analogy with Couette flow of a viscous fluid) which vanishes on \(r = R\), i.e. \(v = -\frac{3}{2} (r - R)^2 - b(r - R)\). We then get \(\frac{dv}{dr} = -a(r - R) - b\), \(\frac{d^2v}{dr^2} = -a\) and the previous equation becomes 
\[-a - a(1 - \frac{R}{r}) - b = -\frac{aR - b}{r} = -3 \dot{\varepsilon}_0 \frac{P}{\sigma_0} + \frac{\sqrt{3} \dot{\varepsilon}_0}{r}\]
which yields
\[
\begin{cases}
  a = \frac{3}{2} \frac{\dot{\varepsilon}_0}{\sigma_0} \frac{P}{\sigma_0} \\
  b = aR - \sqrt{3} \dot{\varepsilon}_0 = \dot{\varepsilon}_0 \left( \frac{3}{2} \frac{PR}{\sigma_0} - \sqrt{3} \right) .
\end{cases}
\] (19)

Therefore, in the visco-plastic zone \(v(r)\) becomes
\[
v(r) = -\frac{3}{4} \dot{\varepsilon}_0 \frac{P}{\sigma_0} (r - R)^2 - \dot{\varepsilon}_0 \left( \frac{3}{2} \frac{PR}{\sigma_0} - \sqrt{3} \right) (r - R)
\] (20)

We can now find the internal radius of the visco-plastic zone. This radius corresponds to a point where \(\sigma_{eq} = \sigma_0\), which implied that \(d_{rz} = 0\) and \(\frac{dv}{dr}(R_0) = 0\) which at it turn gives the equality 
\[-a(R_0 - R) - b = 0\]. With this we have
\[
R_0 = R - \frac{b}{a} = R + \frac{\dot{\varepsilon}_0 \left( \frac{3}{2} \frac{PR}{\sigma_0} - \sqrt{3} \right)}{\frac{3}{2} \dot{\varepsilon}_0 \frac{P}{\sigma_0}} = \frac{2 \sigma_0}{\sqrt{3} P}
\] (21)

Note that \(R_0 = \frac{2 \sqrt{3} \sigma_0}{P}\) is indeed the internal radius since \(\frac{dv}{dr}\) is a decreasing function of \(r\), which vanishes for \(r = R_0\). We then have for \(r > R_0\), \(\frac{dv}{dr} < 0\), this implies that:
\[
\sigma_{rz} = \sigma_0 \left( \frac{1}{3 \dot{\varepsilon}_0} \frac{dv}{dr} - \frac{1}{\sqrt{3}} \right) < \frac{\sigma_0}{\sqrt{3}}
\] (22)

With this \(\sigma_{eq} = -\sqrt{3} \sigma_{rz} > \sigma_0\). The region \(r < R_0\) is in the contrary rigid: it it where visco-plastic we would have \(\frac{dv}{dr} > 0\) which should \(\sigma_{eq}\) by the same reasoning.

In order for material to be visco-plastic, the internal radius \(R_0\) of the visco-plastic zone must be smaller the radius if the tube \(R\), i.e \(\frac{2 \sqrt{3} \sigma_0}{P} < R\) which implies \(P > \frac{2 \sqrt{3} \sigma_0}{R}\). There is no material flow whenever \(P < \frac{2 \sqrt{3} \sigma_0}{R}\); all the material remains rigid. In the case where there is a material flow \((P > \frac{2 \sqrt{3} \sigma_0}{R}\), the velocity
Fig. 2. Velocity field as a function of the radius for different strain regimes of the material.

Note that we recovered the parabolic velocity profile of the well known Couette flow for a viscous fluid by letting the yield limit $\sigma_0$ tending toward 0, the “viscosity coefficient” $\sigma_0/\dot{\varepsilon}_0$ being kept constant.

To find the full solution of the model problem, let us completely determine the stress field starting with the visco-plastic region. The pressure has already been determined with additional additive constant: $p = -P_z + C$. As for the deviatomic part of the stress, we can calculated $\sigma_{rz}$ from the relation given the stress field as a function of the velocity

$$\sigma_{rz} = \sigma_0 \left( \frac{1}{3\dot{\varepsilon}_0} \frac{dv}{dr} - \frac{1}{\sqrt{3}} \right) = \sigma_0 \left( \frac{1}{3\dot{\varepsilon}_0} (-a(r - R) - b) - \frac{1}{\sqrt{3}} \right) = \frac{-P_r}{2} \quad (23)$$

In the rigid region, we introduce the assumption that only the non-zero component of $s$ is $\sigma_{sz}$ only depends on $r$, as in the visco-plastic region. The balance equations read
which implies that

\[ \begin{align*}
\frac{\partial p}{\partial r} &= 0 \\
\frac{\partial \sigma_{rz}}{\partial r} - \frac{dp}{dz} + \frac{\sigma_{rz}(r)}{r} &= 0
\end{align*} \] (24)

which implies that

\[ \begin{align*}
\frac{\partial p}{\partial z} &= P' \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}(r)}{r} &= P'
\end{align*} \] (25)

where \( P' \) is a new constant ("a priori" different \( P \)). Integrating the second equation, we get

\[ \frac{\partial \sigma_{rz}}{\partial r} + \sigma_{rz}(r) = \frac{d}{dr}(r\sigma_{rz}) = -P' \] (26)

which implies that

\[ r\sigma_{rz} = -P'r^2 + A \] (27)
and

$$\sigma_{rz} = -P' \frac{r}{2} + \frac{A}{r}$$

(28)

As $\sigma_{rz}$ can not vanish at infinity at 0, we necessary have $A = 0$, and thus $\sigma_{rz} = -P' \frac{r}{2}$. The continuity of $\sigma_{rz}$ in $r = R_0$ then implies $P = P'$. The formula

$$\sigma_{rz} = -P' \frac{r}{2}$$

(29)

is then valid everywhere in the tube.

Note that $\sigma_{rz}$ is independent of the yield $\sigma_0$: the distribution of the stresses is exactly the same as for a Newtonian viscous fluid.

5 APPLICATION OF THE NORTON VISCOUS MODEL: PROBLEM OF FLOW OF A HOLLOW SPHERE UNDER HYDROSTATIC LOADS.

In this section, we address the scenario of a hollow sphere subject to hydrostatic loading conditions. We assume that the sphere’s material matrix follows the Norton visco-plastic rigid behavior model without yield, effectively neglecting elasticity. This problem’s solution is of interest because it provides insights that can be valuable in defining a homogenized plastic porous material at high temperatures.

Just like with plastic materials, deriving an exact solution for arbitrary loads (where $D$ is arbitrary) poses significant challenges. While it is possible to find an approximate solution, it first necessitates the development of an approximation technique in visco-plasticity, a process analogous to the “approche par l’exterieur” in plasticity.

In this work, we will focus exclusively on the specific case of purely hydrostatic loading, represented by $D = D_m 1$, where $D_m > 0$ indicates the sphere’s expansion. In such a case, with loads exhibiting spherical symmetry, it is feasible to arrive at an analytical solution.

Due to the incompressibility and the spherical symmetry of the problem, the velocity field can be written as:

$$v(\mathbf{v}) = \frac{\alpha}{r^2} \mathbf{e}_r, \quad \alpha = b^3 D_m$$

(30)

As a result the non-zero components of the strain rate and the equivalent strain rate can be found as:
\[
\begin{align*}
D_{rr} &= \frac{dv_r}{dr} = -2D_m \frac{b^3}{r^3} \\
D_{\theta\theta} &= D_{\phi\phi} = \frac{v_r}{r} = D_m \frac{b^3}{r^3} 
\end{align*}
\] (31)

The equivalent strain rate is then defined as:

\[
D_{eq} = \left(D_{rr}^2 + D_{\theta\theta}^2 + D_{\phi\phi}^2\right)^{\frac{1}{2}} = |D_{rr}| = 2D_m \frac{b^3}{r^3}
\] (32)

By inverting the flow rule, we get

\[
D = \frac{3}{2} \frac{\varepsilon_0}{\sigma_0} \left(\frac{\sigma_{eq}}{\sigma_0}\right)^n \frac{s}{\sigma_{eq}}
\] (33)

which implies that:

\[
D_{eq} = \varepsilon_0 \left(\frac{\sigma_{eq}}{\sigma_0}\right)^n ;
\] (34)

the verification of the latter formula is easy as the flow rule was developed such that this relation is automatically satisfied and the colinearity between D and s.

As a consequence, we get

\[
\sigma_{eq} = \sigma_0 \left(\frac{d_{eq}}{\varepsilon_0}\right)^{\frac{1}{n}}
\] (35)

which yields:

\[
s = \frac{2}{3} \frac{d}{\varepsilon_0} \sigma_0 \left(\frac{d_{eq}}{\varepsilon_0}\right)^{\frac{1-n}{n}}
\] (36)

which finally gives

\[
s = \frac{2}{3} \sigma_0 \left(\frac{d_{eq}}{\varepsilon_0}\right)^{\frac{1}{n}} \frac{s}{d_{eq}}
\] (37)

Form there we can deduce the components of the deviatoric tensor s read:
\[
\begin{align*}
\left\{ \begin{array}{c}
s_{rr} &= \frac{2}{3} \sigma_0 \left( \frac{2 D_m b^3}{\dot{\varepsilon}_0 r^3} \right)^{1/n} \frac{d_{rr}}{|d_{rr}|} = -\frac{2}{3} \sigma_0 \left( \frac{2 D_m b^3}{\dot{\varepsilon}_0 r^3} \right)^{1/n} \\
\end{array} \right. \\
\end{align*}
\]  

(38)

\[
\begin{align*}
s_{\theta\theta} = s_{\phi\phi} &= -s_{rr}/2 = \left( \frac{2 D_m b^3}{\dot{\varepsilon}_0 r^3} \right)^{1/n} \\
\end{align*}
\]

(39)

Given the spherical symmetry of the system, the macroscopic stress tensor adopts a hydrostatic form: \( \Sigma = \Sigma_m 1 \). Our next step is to derive an expression for \( \Sigma_m \), leveraging Hill (1967)’s theory and the Mandel (1964)’s lemma. Notably, the virtual velocity field, denoted by \( v^* \), in this scenario aligns with the real velocity field:

\[
v^*(x) = D^*_m \frac{b^3}{r^3}
\]

(39)

which implies

\[
D^*_{rr} = -2 D^*_m \frac{b^3}{r^3}, \quad D^*_{\theta\theta} = D^*_m \frac{b^3}{r^3}.
\]

(40)

As a result, we get:

\[
\begin{align*}
\Sigma : D^* &= \Sigma_m D^*_m 1 : 1 = 3\Sigma_m D^*_m \\
&= (1 - f) \langle \sigma : D^*(x) \rangle_{\Omega - \omega} \\
&= (1 - f) \langle \sigma_{rr}(x) D^*_m(x) + 2\sigma_{\theta\theta}(x) D^*_{\theta\theta}(x) \rangle_{\Omega - \omega} \\
&= (1 - f) 2 D^*_m \left\langle \frac{(\sigma_{rr} - \sigma_{\theta\theta})(x)}{r^3} \right\rangle_{\Omega - \omega} \\
&= 2 \left( 1 - \frac{a^3}{b^3} \right) D^*_m \frac{b^3}{3 \pi (b^3 - a^3)} \int_a^b \frac{(\sigma_{rr} - \sigma_{\theta\theta})(r)}{r^3} 4\pi r^2 d r \nda
&= 3 D^*_m \int_a^b 2(\sigma_{rr} - \sigma_{\theta\theta})(r) \frac{d r}{r}
\end{align*}
\]

(41)

and the latter equation implies that

\[
\Sigma_m = \int_a^b 2(\sigma_{rr} - \sigma_{\theta\theta})(r) \frac{d r}{r}
\]

(42)
As in the case with a cylindrical shape, this results can be written as

\[ \Sigma_m = \int_a^b \frac{d\sigma_{rr}}{dr}(r) = \sigma_{rr}(b) \]  

(43)

since \( \sigma_a = 0 \). Note that this result is intuitively of satisfaction (even though it is not of entire satisfaction for the effective calculation of \( \Sigma_m \)). From our previous calculations:

\[ \sigma_{\theta\theta} = \sigma_{rr} = s_{\theta\theta} = \sigma_0 \left( \frac{2D_m b^3}{\dot{\varepsilon}_0 r^3} \right)^{1/n} \]  

(44)

and thus:

\[
\begin{aligned}
\left\{ 
\Sigma_m &= 2\sigma_0 \left( \frac{2D_m b^3}{\dot{\varepsilon}_0 r^3} \right)^{1/n} \int_a^b r^{-\frac{3}{n}} \frac{dr}{r} = \frac{2}{3} \sigma_0 \left( \frac{2D_m b^3}{\dot{\varepsilon}_0} \right)^{1/n} \int_a^b \frac{dr^3}{r^3} \left( \frac{1}{n+1} \right) \\
&= \frac{2}{3} \sigma_0 \left( \frac{2D_m b^3}{\dot{\varepsilon}_0} \right)^{1/n} (-n) \left( b^{-\frac{3}{n}} - a^{-\frac{3}{n}} \right) ,
\end{aligned}
\]  

(45)

or by inverting we get

\[ D_m = \frac{\dot{\varepsilon}_0}{2} \left( \frac{3}{2n} \right)^{1/n} \left( \frac{\Sigma_m}{\sigma_0} \right) \left( \frac{1}{f} \right) \]

(46)

Above, it was observed that the visco-plastic Norton model, devoid of yield, converges to the plastic Von Mises model as the parameter \( n \) approaches infinity. In this asymptotic scenario, we are compelled to rediscover the expression for the yield limit of the plastic hollow sphere, given by

\[ 2\sigma_0 \ln \frac{2}{b} = -\frac{2}{3} \sigma_0 \ln f. \]

It is noteworthy that this expression was originally derived for an internal pressure applied at \( r = a \), in contrast to the current scenario where an external tension is applied at \( r = b \). Nevertheless, our findings effortlessly demonstrate that the limit loads remain consistent for both loading conditions.

Subsequently, let us analyze the limit of the expression \( n \left( f^{1/n} - 1 \right) \) as \( n \) tends
towards infinity:
\[
n \left( f^{-1/n} - 1 \right) = n \left( e^{-\frac{1}{n}} - 1 \right) = n \left( 1 - \frac{1}{n} \ln f - 1 + O \left( \frac{1}{n^2} \right) \right) \to -\ln f
\]

In the same limit, as \( n \to \infty \), we observe that \( \left( 2 \frac{D_m}{\varepsilon_0} \right)^{1/n} \to 1 \). Consequently, for the limit \( n \to \infty \), we find:
\[
\Sigma_m \to -\frac{2}{3} \sigma_0 \ln f
\]

This result aligns with our expectations.
6 Conclusion

In conclusion, this exploration into the application of viscous models delves into two intricate mechanical behavior challenges. The analysis of the hydrostatic flow of a hollow sphere, employing the Norton viscous model, provides valuable insights for characterizing plastic porous materials at elevated temperatures. Additionally, the investigation of toothpaste flow within a cylindrical tube, utilizing an elasto-visco-plastic model, not only offers a comprehensive solution to the model but also reveals an intriguing correspondence with Newtonian viscous fluid laws. This underscores the chosen visco-elastic model’s applicability and relevance in capturing essential aspects of the intricate dynamics of toothpaste flow.
References
