ANALYSIS QUAL: SEPTEMBER 21, 2023

Please be reminded that to pass the exam you need to show mastery of both real and complex analysis. Please choose at most 10 questions to answer, including at least 4 from problems 1-6 and 4 from problems 7-12. On the front of your paper indicate which 10 problems you wish to have graded.

Problem 1. Prove that finite linear combinations of functions from the family

$$\Big\{x \mapsto \frac{b}{(x-a)^2+b^2} : a \in \mathbb{R} \text{ and } b > 0\Big\}$$

are dense in $L^1(\mathbb{R})$.

Problem 2. Let E denote the set of real numbers in $[0,1]$ without the digit 9 in their decimal expansion, that is, $x \in E$ if it admits the representation

$$x = \sum_{n \geq 0} \frac{a_n}{10^n} \quad \text{with} \quad a_n \in \{0,1,2,3,4,5,6,7,8\}.$$

(a) Show that E is a Borel set.

(b) Show that E has Lebesgue measure zero.

Problem 3. Let U and V be closed subspaces of a Hilbert space $\mathcal{H}$ over $\mathbb{R}$ so that

$$\sup\big\{\langle u, v\rangle : u \in U \text{ and } v \in V \text{ are unit vectors}\big\} < 1.$$

Define

$$W = \{u + v : u \in U \text{ and } v \in V\}.$$

(a) Show that each $w \in W$ admits a unique decomposition $w = u + v$ with $u \in U$ and $v \in V$.

(b) Show that the set W is closed in $\mathcal{H}$.

(c) Show that there is a bounded linear map $T : W \to U$ so that

$$w - T(w) \in V \quad \text{for all} \quad w \in W.$$

Problem 4. For $f \in C^1([0,1];\mathbb{R})$, we define

$$E(f) := \int_0^1 \big(|f'(x)|^2 + |f(x)|^6 - |f(x)|^4\big)\, dx.$$

(i) Show that

$$E_{\min} = \inf_{f \in C^1([0,1];\mathbb{R})} E(f) > -\infty.$$

(ii) Show that if $f_n \in C^1([0,1];\mathbb{R})$ is a minimizing sequence, that is, $E(f_n) \to E_{\min}$ as $n \to \infty$, then the sequence $\{f_n\}$ admits a subsequence that converges in the space $C([0,1];\mathbb{R})$.

Problem 5. Let $\omega : \mathbb{R} \to (0, \infty)$ be a locally integrable function to which we associate a Borel measure via
$$\omega(E) = \int_E \omega(x)\, dx.$$
Let M denote the Hardy–Littlewood maximal function:
$$(Mf)(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)|\, dy.$$
Assume that the function $\frac{1}{\omega}$ is locally integrable and that there exists $C > 0$ so that
$$\omega(\{x \in \mathbb{R} : |(Mf)(x)| > \lambda\}) \le \frac{C}{\lambda^2} \int_{\mathbb{R}} |f(x)|^2 \omega(x)\, dx$$
uniformly in $\lambda > 0$ and functions $f : \mathbb{R} \to \mathbb{R}$ for which the right-hand side above is finite. Prove that
$$\sup_{x\in\mathbb{R}, r>0} \Big(\frac{1}{2r} \int_{x-r}^{x+r} \omega(y)\, dy\Big)\Big(\frac{1}{2r} \int_{x-r}^{x+r} \frac{1}{\omega(y)}\, dy\Big) < \infty.$$
Hint: Apply the hypothesis to a well chosen function f and constant λ.

Problem 6. Consider the following sequence of functions:
$$f_n : \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = \frac{\sin(n^4 x)}{n^3 x}$$
(a) Prove that f_n does *not* converge to zero in $L^4(\mathbb{R})$.
(b) Prove that f_n does converge to zero weakly in $L^4(\mathbb{R})$, that is, for any $\phi \in L^{4/3}(\mathbb{R})$ we have
$$\int_{\mathbb{R}} f_n(x)\phi(x)\, dx \to 0 \qquad \text{as } n \to \infty.$$

Problem 7. Show that for every real number x that is not an integer, the series
$$\sum_{n=1}^{\infty} \frac{1}{x^2 - n^2}$$
is absolutely convergent, and sums to $\frac{\pi \cot(\pi x)}{2x} - \frac{1}{2x^2}$.

Problem 8. For z_1, z_2 in the unit disk $D(0,1) := \{z \in \mathbb{C} : |z| < 1\}$, define the quantity
$$\Delta(z_1, z_2) := \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|.$$
(i) Let $\alpha \in D(0,1)$, and let $g : D(0,1) \to D(0,1)$ denote the Möbius transformation
$$g(z) := \frac{z - \alpha}{1 - \overline{\alpha} z}.$$
Show that $\Delta(g(z_1), g(z_2)) = \Delta(z_1, z_2)$ for all $z_1, z_2 \in D(0,1)$.
(ii) If $f : D(0,1) \to D(0,1)$ is holomorphic and z_1, z_2 are elements of $D(0,1)$, establish the inequality
$$\Delta(f(z_1), f(z_2)) \le \Delta(z_1, z_2).$$
(iii) Determine when equality occurs in (ii).

Problem 9.

(i) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, and suppose that there exist two complex numbers ω_1, ω_2, linearly independent over the reals, as well as complex constants c_1, c_2, such that $f(z + \omega_1) = f(z) + c_1$ and $f(z + \omega_2) = f(z) + c_2$ for all complex numbers z. Show that f is a linear polynomial, that is to say there exist complex numbers a, b such that $f(z) = az + b$.

(ii) For any real number a, let R_a denote the closed rectangle

$$R_a := \{x + iy : 0 \leq x \leq a; 0 \leq y \leq 1\}.$$

Suppose that there is a homeomorphism $\phi : R_a \to R_b$ which is holomorphic on the interior of R_a, and maps each of the four sides of R_a to the corresponding side of R_b (for instance, ϕ maps the right side $\{a + iy : 0 \leq y \leq 1\}$ of R_a to the right side $\{b + iy : 0 \leq y \leq 1\}$ of R_b). Show that $a = b$. (*Hint:* is there a way to somehow enlarge the domain of ϕ? You may find part (i) to be useful.)

Problem 10. Let $f : \mathbb{R} \to \mathbb{C}$ be a smooth function with compact support, and let $F : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ be the Cauchy integral

$$F(z) := \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(y)\ dy}{y - z}.$$

(i) Explain why F is holomorphic on $\mathbb{C}\backslash\mathbb{R}$.

(ii) For any real number x, show that the principal value integral

$$p.v. \int_{-\infty}^{\infty} \frac{f(y)\ dy}{y - x} := \lim_{\varepsilon \to 0^+} \int_{|y-x| \geq \varepsilon} \frac{f(y)\ dy}{y - x}$$

exists, and establish the *Sokhotski–Plemelj formulae*

$$\lim_{\varepsilon \to 0^+} F(x + i\varepsilon) = \frac{1}{2} f(x) + \frac{1}{2\pi i} p.v. \int_{-\infty}^{\infty} \frac{f(y)\ dy}{y - x}$$

and

$$\lim_{\varepsilon \to 0^+} F(x - i\varepsilon) = -\frac{1}{2} f(x) + \frac{1}{2\pi i} p.v. \int_{-\infty}^{\infty} \frac{f(y)\ dy}{y - x}.$$

Problem 11. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function which is not a polynomial. Show that the expression

$$\frac{1}{\log R} \int_0^{2\pi} \max(\log |f(Re^{i\theta})|, 0)\ d\theta$$

diverges to infinity as $R \to +\infty$.

Problem 12. Show that the improper Fourier integral

$$p.v. \int_{-\infty}^{\infty} \frac{\sin(x)}{x} e^{i\xi x}\ dx = \lim_{\varepsilon \to 0^+, R \to +\infty} \int_{\varepsilon \leq |x| \leq R} \frac{\sin(x)}{x} e^{i\xi x}\ dx$$

is equal to π when ξ is a real number with $|\xi| < 1$, and vanishes when ξ is a real number with $|\xi| > 1$.