Applied Differential Equations

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper.
Write your university identification number at the top of each sheet of paper.

DO NOT WRITE YOUR NAME!
Complete this sheet and staple to your answers. Read the directions of the exam very carefully.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE
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1. ______________________	5. __________________________
2. ______________________	6. __________________________
3. ______________________	7. __________________________
4. ______________________	8. __________________________

Pass/fail recommend on this form.

Total score: ___________________

ADE Exam, Spring 2024
Department of Mathematics, UCLA

1. [10 points] In 1991, John Tyson published a model of the cell-division cycle that is based on the interaction between two proteins. As part of his paper, Tyson derived the following dimensionless dynamical system:

$$\dot{u} = b(v-u)(\alpha+u^2) - u\,,$$
$$\dot{v} = c - u\,, \tag{1}$$

where $u \geq 0$ is proportional to the concentration of one protein and $v \geq 0$ is proportional to the concentration of the other protein. The positive parameters $b \gg 1$ and $\alpha \ll 1$ are fixed, and they satisfy $8\alpha\beta < 1$. The parameter $c > 0$ is adjustable.

   (a) Sketch the nullclines of (1).

   (b) An isolated period orbit (i.e., a limit cycle) experiences a so-called *relaxation oscillation* when a very slow build-up alternates with a very fast discharge.

   When $8\alpha\beta \ll 1$, show that the dynamical system (1) has relaxation oscillations when $c_1 < c < c_2$, where you will determine c_1 and c_2 approximately by exploiting the fact that $8\alpha\beta \ll 1$.

   [In part (b), you may use the approximation $8\alpha\beta \ll 1$ directly by dropping small terms (i.e., without doing any formal asymptotic analysis). The $8\alpha\beta \ll 1$ approximation in part (b) is to be able to analytically approximate c_1 and c_2. In fact, there are relaxation oscillations in the full parameter range that is specified in the problem.]

   (c) Show that the dynamical system (1) is "excitable" if c is slightly less than c_1.

   *[Note: A system is **excitable** if (i) it has a globally attracting equilibrium point, but (ii) certain perturbations can send it on a long excursion through phase space before it returns to the equilibrium point.]*

2. [10 points] By using and analyzing Frobenius series about the point $x = 0$, find two linearly independent solutions of the Bessel equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0\,, \tag{2}$$

where ν is a real number.

3. [10 points]

(a) By explicit calculation, show that the Green's function $G(\mathbf{x})$ for Laplace's equation in $\mathbb{R}^3$, satisfying

$$-\Delta G = \delta(\mathbf{x})$$

with $G \to 0$ as $|\mathbf{x}| \to \infty$, is

$$G(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}\,.$$

[You may assume that the Green's function is unique.]

(b) The Green's function singularity is located at a point $\boldsymbol{\xi}$ within a wedge defined in cylindrical (r, θ, z) coordinates by: $r > 0$, $0 < \theta < \alpha$, $-\infty < z < \infty$. We impose $u = 0$ on both $\theta = 0$ and $\theta = \alpha$. Derive and justify conditions on the wedge angle, α, to ensure the complete Green's function can be written as a finite sum of terms of the form $\pm G(\mathbf{x}-\boldsymbol{\xi}_i)$, with G of the form given in (a).

4. [10 points] Let $D \subset R^n$ be a bounded open set, and let ∂D be its boundary. Consider the eigenvalue problem

$$-\Delta u = \lambda u\,, \tag{3}$$

where $u \in \mathcal{C}^2(D) \cup \mathcal{C}^1(\overline{D})$ and, for all $x \in \partial D$, either $u = 0$ or $\dfrac{\partial u}{\partial n} = 0$, except possibly on a measure-0 set.

(a) Show that all eigenvalues are nonnegative.

(b) Show that the eigenfunction corresponding to the smallest eigenvalue minimizes $\int_D \|\nabla u\|^2\, dV$ among all functions with $\int_D u^2\, dV = 1$ that satisfy the requisite boundary conditions.

(c) Using the minimum principle that you derived in part (b), along with a suitable test function, estimate the smallest eigenvalue when $D = \{(x, y) : 0 < x < 1\,, 0 < y < 1\}$ and $u = 0$ on ∂D.

5. [10 points] Suppose that $u(x,t)$ satisfies

$$u_t = \Delta u + u(1-u) \tag{4}$$

on the bounded open set $D \subset \mathbb{R}^n$ with boundary ∂D. Assume that $u \in \mathcal{C}^2(D) \cup \mathcal{C}^1(\overline{D})$ and that $u = 0$ on ∂D.

If $u(x,0) < 1$, show that $u(x,t) < 1$ for all t.

6. [10 points] Consider the heat equation

$$u_t = u_{xx}$$

on the real line with initial data

$$u_0 = \begin{cases} -1\,, & x < 0 \\ +1\,, & x \geq 0\,. \end{cases}$$

(a) Show that the solution $u(x,t)$ satisfies $\lim_{t\to\infty} u(x,t) = 0$.

(b) Is the $t \to \infty$ limit uniform in x? Prove your answer.

7. [10 points] Consider the Cauchy problem

$$u_t + u_x = u^2$$

with initial condition

$$u(x,0) = \frac{1}{1+x^2}\,.$$

(a) Compute the smooth solution $u(x,t)$ explicitly for $0 < t < 1$.

(b) Show that the solution blows up as $t \to 1$.

(c) Where does the blowup occur in space?

8. [10 points] Consider the damped wave equation

$$\epsilon^2 u_{tt} + u_t = \Delta u$$

in a bounded region $D \in R^3$ with the boundary condition

$$u(x,t) = 0\,, \quad x \in \partial D\,.$$

(a) Show that the energy

$$E(t) = \int_D \left[\epsilon^2 u_t^2 + |\nabla u|^2\right] dx$$

is nonincreasing in time.

(b) Consider an ϵ-dependent family of solutions that satisfy the initial conditions

$$u^\epsilon(x,0) = 0\,,$$
$$u_t^\epsilon(x,0) = \epsilon^{-\alpha} f(x)$$

for some fixed $\alpha \in (0,1)$.

Use the result of part (a) to show that $\int_D |\nabla u(x,t)|^2\, dx \to 0$ as $\epsilon \to 0$ uniformly on $[0,T]$ for any $T > 0$.