ALGEBRA QUALIFYING: WINTER 2024

Test instructions:

Write your UCLA ID number on the upper right corner of *each* sheet of paper you use. Do not write your name anywhere on the exam.

Clearly indicate, by circling the number of the problem on the front page, which 8 of the 10 problems you want us to grade.

No books, notes, calculators, computers or other printed or electronic materials can be used on the exam.

Please staple your problems in the order they are listed in the exam.

1	2	3	4
5	6	7	8
9	10		

Problem 1. Let α be a complex root of $x^6 + 3$. Set $K = \mathbb{Q}(\alpha)$.

(1) Show that the extension $K \supset \mathbb{Q}$ is normal.
(2) Compute the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

Problem 2. Prove the two following statements.
Every maximal ideal of the ring $\mathbb{C}[x, y]$ has the form $\langle x - \alpha, y - \beta\rangle$ with $\alpha, \beta \in \mathbb{C}$.
Every maximal ideal of the ring $\mathbb{R}[x, y]$ is either of the form:

(1) $\langle x - \alpha, y - \beta\rangle$ with $\alpha, \beta \in \mathbb{R}$, or
(2) $\langle \ell, q\rangle$ where $\ell \in \mathbb{R}[x, y]$ is linear and q is an irreducible polynomial of degree 2 on either x or y.

Problem 3. Find all positive integers n such that $\cos(2\pi/n)$ is a rational number.

Problem 4. Let R be a Noetherian ring. Let I and J be two ideals of R. Show that

$$\mathrm{Tor}_1^R(R/I, R/J) \simeq (I \cap J)/IJ.$$

Problem 5. Prove that a finitely generated projective module M over a local ring $(R, \mathfrak{m})$ is free.

Problem 6. Let G be a finite group of order 300. Show that G is not simple by considering its action on its Sylow 5-subgroups.

Problem 7. Let G be a group and H a subgroup of G.

(a). If G is nilpotent, is H nilpotent?
(b). If H is normal in G and G is nilpotent, is G/H nilpotent?
(c). If H is normal in G and H and G/H are nilpotent, is G nilpotent?

Problem 8. (a). Let A be a finite abelian group and χ a complex character of A. Show that

$$\sum_{a \in A} |\chi(a)|^2 \geq |A| \cdot \chi(1).$$

(b). Let G be a finite group and A an abelian subgroup of G of index n. Let ψ be a complex irreducible character of G. Show that $\psi(1) \leq n$ (apply part (a) to the restriction of ψ to A).

Problem 9. Let A be a finite-dimensional algebra over an algebraically closed field k. Recall that the Jacobson radical $J(R)$ of a left Artinian ring R is a nilpotent ideal and $R/J(R)$ is a semi-simple ring.

Show that the following assertions are equivalent:

(1) The simple A-modules are 1-dimensional
(2) $J(A)$ is the set of nilpotent elements of A.

Problem 10. Let F be a functor from a small category I to the category of abelian groups. Show that F admits a colimit by constructing it as a quotient of $\bigoplus_i F(i)$, where i runs over the set of objects of I.