# Analysis

## Read the instructions of the exam carefully.

Complete this sheet and staple to your answers.

STUDENT ID NUMBER ____________________________________________

DATE: ________________________________________________________

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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**Pass/Fail recommendation on this form.**

Total score: ______________________

Form revised 3/08

## Analysis Qualifying Exam, March 28, 2024

**Rules:** Solve no more than ten problems. You must demonstrate adequate knowledge of both real analysis (Problems 1–6) and complex analysis (Problems 7–12). Parts of a problem may not carry equal weight.

**Problem 1**. For each $n \in \mathbb{N}$, let $a_n : \mathbb{R} \to [0, \infty)$ be a Borel-measurable function. Show that

$$\left\{(x, y) \in \mathbb{R} \times [0, \infty) : \sum_{n=1}^{\infty} a_n(x) y^n < \infty\right\}$$

is Borel-measurable.

**Problem 2**. Let $K \subset \mathbb{R}$ be a compact set of positive Lebesgue measure, that is, $|K| > 0$. For each $n \in \mathbb{N}$, we define sets $K_n$ and Borel measures $\mu_n$ as follows:

$$K_n := \left\{x \in \mathbb{R} : \operatorname{dist}(x, K) \le \tfrac{1}{n}\right\} \quad \text{and} \quad \mu_n(A) = \frac{|A \cap K_n|}{|K_n|}.$$

Suppose $\mu_n \to \mu$ in the weak-$*$ topology. Show that $|\operatorname{supp}(\mu)| = |K|$.

**Problem 3**. Fix $f \in L^3(\mathbb{R}^2)$ with respect to Lebesgue measure. Show that

$$f_n(x, y) := \int_0^1 \int_0^{2\pi} f\Big(x + \frac{r\cos(\theta)}{n}, y + \frac{r\sin(\theta)}{n}\Big)\big[1 - 2r\big]\, d\theta\, dr$$

converges to zero as $n \to \infty$ in the following two ways:
(a) almost everywhere.
(b) in $L^3(\mathbb{R}^2)$.

**Problem 4**. Fix $f \in L^1(\mathbb{R})$ that is non-negative and satisfies $\int f(x)\, dx = 1$. We then define the $n$-fold convolution

$$f_n(x) := \int \cdots \int f(x - y_1 - y_2 - \cdots - y_n) f(y_1) f(y_2) \cdots f(y_n)\, dy_1\, dy_2 \cdots dy_n$$

of $f$ with itself. Show that the sequence $f_n(x)$ does *not* converge in $L^1(\mathbb{R})$.

**Problem 5**. Fix $1 \le p < q < \infty$. Throughout this problem, $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$ are defined using Lebesgue measure and $|A|$ denotes the Lebesgue measure of the set $A$.
(a) Suppose $f \in L^p(\mathbb{R})$ satisfies $\int_A |f(x)|^q\,dx < \infty$ for every Borel subset $A$ with $|A| = 1$. Show that $f \in L^q(\mathbb{R})$.
(b) Show that there exists $f \in L^p(\mathbb{R})$ so that $\int_a^{a+1} |f(x)|^q\,dx < \infty$ for every $a \in \mathbb{R}$ but $f \notin L^q(\mathbb{R})$.

**Problem 6**. Let $\mathcal{H}$ be a Hilbert space and $U : \mathcal{H} \to \mathcal{H}$ a unitary operator, that is, $U$ is bounded, linear, and invertible with the inverse equal to its adjoint $U^*$.
(a) Prove that $\mathrm{Ker}(U - I)^\perp = \overline{\mathrm{Ran}(U - I)}$, where $\mathrm{Ran}(U - I)$ denotes the range of $U - I$ and $I$ is the identity operator on $\mathcal{H}$.
(b) Let $P$ denote the (orthogonal) projection of $\mathcal{H}$ onto $\mathrm{Ker}(U - I)$. Prove that for any vector $v \in \mathcal{H}$,

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k v \longrightarrow Pv \quad \text{in the } \mathcal{H}\text{-norm, as } n \to \infty.$$

**Problem 7**. Rigorously evaluate

$$\int_{-\infty}^{\infty} \frac{\log|x+i|}{x^2+4}\,dx.$$

**Problem 8**. For each $n \in \mathbb{N}$, suppose $f_n : \mathbb{D} \to (-1, 1)$ is harmonic. (Here $\mathbb{D}$ denotes the unit disk in the complex plane.)
(a) Show that there is a subsequence of the functions $f_n$ that converges uniformly on compact subsets of $\mathbb{D}$.
(b) Suppose $f(z)$ is such a subsequential limit and that $f(0) = 1$. Show that $f(z) = 1$ for all $z \in \mathbb{D}$.

**Problem 9**. Find all entire functions $f(z)$ with the property that if one writes $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$ and $u, v$ are the real and imaginary parts of $f$, then for all $x, y \in \mathbb{R}$ we have

$$u(x, y) + v(x, y) \le x + y.$$

**Problem 10**. Let $h : (-\infty, 0] \to \mathbb{R}$ be continuous and define

$$\Gamma = \{x + ih(x) : -\infty < x \leq 0\}.$$

Suppose $f : \mathbb{C} \to \mathbb{C}$ is continuous and that $f$ is holomorphic in $\mathbb{C} \setminus \Gamma$. Show that if $f(z) = 0$ for all $z \in \Gamma$, then $f(z) = 0$ for all $z \in \mathbb{C}$.

**Problem 11**. Let $f : \mathbb{C} \to \mathbb{C}$ be the unique holomorphic function with

$$f(0) = 0 \quad \text{and} \quad f'(z) = e^{z^2}.$$

Show that this function admits a convergent representation

$$f(z) = z \exp\bigl(cz^2\bigr) \prod_n \left[\left(1 - \frac{z^2}{z_n^2}\right) \exp\left(\frac{z^2}{z_n^2}\right)\right]$$

for some $c \in \mathbb{R}$ and some (finite or infinite) collection of complex numbers $z_n$.

**Problem 12**. Consider the following polynomial of $z, w \in \mathbb{C}$:

$$P(w, z) := w^3(w-2)^3 + z.$$

(a) Find an explicit $\delta > 0$ so that $w \mapsto P(w, z)$ has precisely three zeros (counted by multiplicity) in the unit disk whenever $|z| < \delta$.
(b) Let us write $w_1(z), w_2(z), w_3(z)$ for these three zeros. Show that

$$z \mapsto w_1(z) + w_2(z) + w_3(z)$$

defines a holomorphic function on $|z| < \delta$.
*Warning*: Each individual $w_i(z)$ will *not* be holomorphic!