Read the instructions of the exam carefully.
Complete this sheet and staple to your answers.

STUDENT ID NUMBER ______________________________________________________
DATE: __________________________________________________________________

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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Pass/Fail recommendation on this form.

Total score: ____________________

Form revised 3/08
Rules: Solve no more than ten problems. You must demonstrate adequate knowledge of both real analysis (Problems 1–6) and complex analysis (Problems 7–12). Parts of a problem may not carry equal weight.

Problem 1. For each \( n \in \mathbb{N} \) let \( a_n : \mathbb{R} \to [0, \infty) \) be a Borel-measurable function. Show that

\[
\left\{ (x, y) \in \mathbb{R} \times [0, \infty) : \sum_{n=1}^{\infty} a_n(x) y^n < \infty \right\}
\]

is Borel-measurable.

Problem 2. Let \( K \subset \mathbb{R} \) be a compact set of positive Lebesgue measure, that is, \( |K| > 0 \). For each \( n \in \mathbb{N} \), we define sets \( K_n \) and Borel measures \( \mu_n \) as follows:

\[
K_n := \left\{ x \in \mathbb{R} : \text{dist}(x, K) \leq \frac{1}{n} \right\} \quad \text{and} \quad \mu_n(A) = \frac{|A \cap K_n|}{|K_n|}.
\]

Suppose \( \mu_n \to \mu \) in the weak-* topology. Show that \( |\text{supp}(\mu)| = |K| \).

Problem 3. Fix \( f \in L^3(\mathbb{R}^2) \) with respect to Lebesgue measure. Show that

\[
f_n(x, y) := \int_0^1 \int_0^{2\pi} f \left( x + \frac{r \cos(\theta)}{n}, y + \frac{r \sin(\theta)}{n} \right) \left[ 1 - 2r \right] d\theta dr
\]

converges to zero as \( n \to \infty \) in the following two ways:

(a) almost everywhere.

(b) in \( L^3(\mathbb{R}^2) \).

Problem 4. Fix \( f \in L^1(\mathbb{R}) \) that is non-negative and satisfies \( \int f(x) \, dx = 1 \). We then define the \( n \)-fold convolution

\[
f_n(x) := \int \cdots \int f(x - y_1 - y_2 - \cdots - y_n) f(y_1) f(y_2) \cdots f(y_n) dy_1 dy_2 \cdots dy_n
\]

of \( f \) with itself. Show that the sequence \( f_n(x) \) does not converge in \( L^1(\mathbb{R}) \).
Problem 5. Fix $1 \leq p < q < \infty$. Throughout this problem, $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$ are defined using Lebesgue measure and $|A|$ denotes the Lebesgue measure of the set $A$.

(a) Suppose $f \in L^p(\mathbb{R})$ satisfies $\int_A |f(x)|^q \, dx < \infty$ for every Borel subset $A$ with $|A| = 1$. Show that $f \in L^q(\mathbb{R})$.

(b) Show that there exists $f \in L^p(\mathbb{R})$ so that $\int_{a}^{a+1} |f(x)|^q \, dx < \infty$ for every $a \in \mathbb{R}$ but $f \notin L^q(\mathbb{R})$.

Problem 6. Let $\mathcal{H}$ be a Hilbert space and $U : \mathcal{H} \to \mathcal{H}$ a unitary operator, that is, $U$ is bounded, linear, and invertible with the inverse equal to its adjoint $U^*$.

(a) Prove that $\text{Ker}(U - I)^\perp = \overline{\text{Ran}(U - I)}$, where $\text{Ran}(U - I)$ denotes the range of $U - I$ and $I$ is the identity operator on $\mathcal{H}$.

(b) Let $P$ denote the (orthogonal) projection of $\mathcal{H}$ onto $\text{Ker}(U - I)$. Prove that for any vector $v \in \mathcal{H}$,

\[ \frac{1}{n} \sum_{k=0}^{n-1} U^k v \longrightarrow P v \quad \text{in the $\mathcal{H}$-norm, as $n \to \infty$.} \]

Problem 7. Rigorously evaluate

\[ \int_{-\infty}^{\infty} \log |x + i| \frac{dx}{x^2 + 4}. \]

Problem 8. For each $n \in \mathbb{N}$, suppose $f_n : \mathbb{D} \to (-1, 1)$ is harmonic. (Here $\mathbb{D}$ denotes the unit disk in the complex plane.)

(a) Show that there is a subsequence of the functions $f_n$ that converges uniformly on compact subsets of $\mathbb{D}$.

(b) Suppose $f(z)$ is such a subsequential limit and that $f(0) = 1$. Show that $f(z) = 1$ for all $z \in \mathbb{D}$.

Problem 9. Find all entire functions $f(z)$ with the property that if one writes $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$ and $u, v$ are the real and imaginary parts of $f$, then for all $x, y \in \mathbb{R}$ we have

\[ u(x, y) + v(x, y) \leq x + y. \]
Problem 10. Let \( h : (-\infty, 0] \to \mathbb{R} \) be continuous and define
\[
\Gamma = \{ x + ih(x) : -\infty < x \leq 0 \}.
\]
Suppose \( f : \mathbb{C} \to \mathbb{C} \) is continuous and that \( f \) is holomorphic in \( \mathbb{C} \setminus \Gamma \). Show that if \( f(z) = 0 \) for all \( z \in \Gamma \), then \( f(z) = 0 \) for all \( z \in \mathbb{C} \).

Problem 11. Let \( f : \mathbb{C} \to \mathbb{C} \) be the unique holomorphic function with
\[
f(0) = 0 \quad \text{and} \quad f'(z) = e^{z^2}.
\]
Show that this function admits a convergent representation
\[
f(z) = z \exp(cz^2) \prod_n \left[ \left( 1 - \frac{z^2}{z_n^2} \right) \exp\left( \frac{z^2}{z_n^2} \right) \right]
\]
for some \( c \in \mathbb{R} \) and some (finite or infinite) collection of complex numbers \( z_n \).

Problem 12. Consider the following polynomial of \( z, w \in \mathbb{C} \):
\[
P(w, z) := w^3(w - 2)^3 + z.
\]
(a) Find an explicit \( \delta > 0 \) so that \( w \mapsto P(w, z) \) has precisely three zeros (counted by multiplicity) in the unit disk whenever \( |z| < \delta \).
(b) Let us write \( w_1(z), w_2(z), w_3(z) \) for these three zeros. Show that
\[
z \mapsto w_1(z) + w_2(z) + w_3(z)
\]
defines a holomorphic function on \( |z| < \delta \).

Warning: Each individual \( w_i(z) \) will not be holomorphic!