Name: ____________________________
Student ID Number: ____________________________

UCLA MATHEMATICS – BASIC EXAM: SPRING 2024

GRADING: The exam is four hours long, and has 12 questions, 6 in analysis and 6 in linear algebra. Each question is worth 10 points. A score of at least 70 points on the exam, with at least 6 near complete (9 or 10 points) answers, including at least 3 in each of the areas (analysis, linear algebra), typically ensures a passing grade.

INSTRUCTIONS: Do any 10 of the following questions. If you attempt more than 10 questions, indicate which ones you would like to be considered for credit (otherwise the first 10 will be taken). Little or no credit will be given for answers without adequate justification. You have 4 hours. Good luck.

NOTATION: We denote by $\mathbb{N} = 1, 2\ldots$ the natural numbers, and by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers, respectively.

#	Score	Counts in 10?
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
Total		10

Linear Algebra

1. Let A be a real $n \times n$ matrix. Suppose that for all invertible real $n \times n$ matrices B, we have $\operatorname{tr}(AB) = 0$. Prove that $A = 0$.

2. Suppose that $\vec{x}(t) \in \mathbb{R}^n$ satisfies the system of differential equations

$$x_1'(t) = x_2(t) - x_1(t), \quad x_2'(t) = x_3(t) - x_2(t), \quad \dots \quad x_n'(t) = x_1(t) - x_n(t),$$

for all $t \geq 0$ with initial condition $\vec{x}(0) = (1, 2, \dots, n)$. Find the limit of $\vec{x}(t)$ as $t \to \infty$.

3. Let V be a finite dimensional inner product space over $\mathbb{C}$. Let L be a normal operator on V. Given a value $z \in \mathbb{C}$ and a unit vector $v \in V$, prove that L has an eigenvalue λ such that:

$$\|Lv - zv\| \geq |\lambda - z|.$$

Here $\|v\|^2 = \langle v, v \rangle$.

4. Let $v = (1, 1, 1)$ and let T be the rotation of angle $\theta \in [0, 2\pi)$ around the axis determined by v. The rotation is counterclockwise as viewed from the point $(10, 10, 10)$ looking toward the origin. Find the matrix representation of T in the standard basis.
Note: You may write your answer as a product of matrices.

5. Let A be a 3×3 real matrix such that $\det(A) > 0$ and $A^t = A^{-1}$. Show that there exists a nonzero vector $v \in \mathbb{R}^3$ such that $Av = v$.

6. Let T be a linear operator on $\mathbb{C}^4$ that satisfies the polynomial identity

$$T^4 + 4T^3 - 16T - 16I = 0,$$

where I denotes the identity operator. Assume that

$$\dim(\operatorname{im}(T + 2I)) = 2 \quad \text{and} \quad \dim(\operatorname{im}(T - 2I)) = 3.$$

Provide the Jordan canonical form for the linear operator T. Justify your answer.

Analysis

7. Fix $g \in C([0,1] \times [0,1])$. For a Riemann integrable function $f : [0,1] \to \mathbb{R}$, we define the operator T via

$$[Tf](x) = \int_0^1 g(x,y)f(y)dy.$$

(a) Prove that the function Tf is continuous on $[0,1]$.

(b) Show that the set

$$S = \big\{Tf \,:\, f \text{ is Riemann integrable and } \sup_{x\in[0,1]} |f(x)| \le 1\big\}$$

is precompact (i.e. its closure is compact) in the space $C([0,1])$ equipped with the uniform metric

$$d_\infty(f_1, f_2) = \sup_{x\in[0,1]} |f_1(x) - f_2(x)|.$$

8. Let $f_n : [0,\infty) \to \mathbb{R}$ be a sequence of functions defined recursively as follows:

$$f_1(t) := 0 \quad \text{and} \quad f_{n+1}(t) = e^{-4t} + \int_0^t f_n(s)e^{-4s}ds \quad \text{for } n \ge 1.$$

Show that $f(t) = \lim_{n\to\infty} f_n(t)$ exists for all $t \ge 0$. Identify the limit function $f(t)$.

9. Fix two real numbers $a_1 > b_1 > 0$ and define sequences $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ via the recurrence relations:

$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \frac{2a_nb_n}{a_n + b_n} \quad \text{for all } n \ge 1.$$

(1) Prove that

$$a_n > a_{n+1} > b_{n+1} > b_n \quad \text{for all } n \ge 1.$$

(2) Conclude that the two sequences converge and show that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \sqrt{a_1b_1}.$$

10. Let E be an uncountable subset of $[0,1]$. Prove that there exist $0 \le \alpha < \beta \le 1$ such that both the sets

$$\{x \in E : x < \gamma\} \quad \text{and} \quad \{x \in E : x > \gamma\}$$

are uncountable *if and only if* $\alpha < \gamma < \beta$.

11. Let $f : \mathbb{Q} \to \mathbb{R}$ be a uniformly continuous function. Prove that the set

$$E = \overline{\{f(x) : x \in \mathbb{Q}\}}$$

is connected. Here $\overline{A}$ denotes the closure of the set A in $\mathbb{R}$.

12. Let $f : [0, 1] \to (0, \infty)$ be a continuous function and let

$$M = \sup_{x \in [0,1]} f(x).$$

Show that

$$\lim_{n \to \infty} \left(\int_0^1 [f(x)]^n \, dx \right)^{\frac{1}{n}} = M.$$