QUALIFYING EXAM
Geometry/Topology
March 2024

Attempt all ten problems. Each problem is worth 10 points. You must fully justify your answers.

1. (a) (5 pts) Show that the Lie group $SL_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \det(A) = 1\}$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.

   (b) (5 pts) Show that the Lie group $SL_2(\mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) \mid \det(A) = 1\}$ is diffeomorphic to $S^3 \times \mathbb{R}^3$.

   (Hint to both parts: normalize the first row vector.)

2. Let
$$\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\})/(x_0, \ldots, x_n) \sim t(x_0, \ldots, x_n),$$
for all $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} - \{0\}$ and $t \in \mathbb{R} - \{0\}$ be the real n-dimensional projective space, and let $X = \{[(x_0, \ldots, x_n)] \in \mathbb{RP}^n \mid x_0 = 0\}$, where $[(x_0, \ldots, x_n)]$ is the equivalence class of $(x_0, \ldots, x_n)$. Is it possible to find a smooth map $f : \mathbb{RP}^n \to \mathbb{R}$ with $0 \in \mathbb{R}$ as a regular value and preimage $f^{-1}(0) = X$?

3. Let $M \subset N$ a compact submanifold of codimension ≥ 3. Show that if N is connected and simply connected, then so is the complement $N - M$.

4. Let M, N be closed oriented n-manifolds with N connected. Show that if $f : M \to N$ has nonzero degree, then $f^* : H^*_{\mathrm{dR}}(N; \mathbb{R}) \to H^*_{\mathrm{dR}}(M; \mathbb{R})$ is injective. (Hint: First show that $f^* : H^n_{\mathrm{dR}}(N; \mathbb{R}) \to H^n_{\mathrm{dR}}(M; \mathbb{R})$ is injective.)

5. Find two vector fields X and Y on $\mathbb{R}^3$ such that X, Y, $[X, Y]$ are everywhere linearly independent.

6. Let X be a topological space and $p \in X$. Let Y be the topological space obtained from $X \times [0, 1]$ by contracting $(X \times \{0, 1\}) \cup (\{p\} \times [0, 1])$ to a point. Describe the relation between the homology groups of X and Y.

7. Exhibit a space whose fundamental group is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$, where $\mathbb{Z}/k\mathbb{Z}$ denotes the integers modulo k and $*$ denotes the free product. Also exhibit a space whose fundamental group is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$.

8. (a) (3 pts) Define what it means for a covering space to be regular.

   (b) (7 pts) Give an example of an irregular covering space of the wedge sum $S^1 \vee S^1$.

9. (a) (2 pts) Show that a nonsingular linear $A : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ induces a smooth map $\Phi_A : \mathbb{CP}^n \to \mathbb{CP}^n$.

   (b) (2 pts) Show that the fixed points of Φ_A correspond to eigenvectors of the original matrix.

(c) (3 pts) Show that Φ_A is a Lefschetz map if the eigenvalues of A all have multiplicity 1.

(d) (3 pts) Show that the Lefschetz number of Φ_A is $n+1$.

10. Consider the following subsets of $\mathbb{R}^3$:

$$A = \{(0,0,z) \mid z \in \mathbb{R}\},$$
$$B = \{(\cos\theta, \sin\theta, 0) \mid \theta \in \mathbb{R}\},$$
$$C = \{(\cos\theta, \sin\theta + 5, 0) \mid \theta \in \mathbb{R}\}.$$

Show that $\mathbb{R}^3 - A - B$ and $\mathbb{R}^3 - A - C$ are not homeomorphic.