Name: ________________________
Student ID Number: ________________________

UCLA MATHEMATICS – NUMERICAL ANALYSIS: SPRING 2024

INSTRUCTIONS and GRADING:
There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score. You have to demonstrate a sufficient amount of work on both groups of problems $[1-4]$ and $[5-8]$ to obtain a passing score.

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM. You have 4 hours. Good luck.

#	Score	Points Possible
1		5
2		5
3		5
4		5
5		10
6		10
7		10
8		10
Total		60

1. Determine the constants a, b, and c so that the quadrature rule:

$$\int_{-1}^{1} f(x)\,\mathrm{dx} \approx a\,f(-1) + b\,f(0) + c\,f(1)$$

has the highest possible degree of precision.

2. Let $x \in \mathbb{R}^n$ solve $Ax = b$ and $x + \Delta x$ solve $A(x + \Delta x) = b + \Delta b$. Show that:

$$\frac{||\Delta x||_2}{||x||_2} \leq \mathrm{cond}(A)\,\frac{||\Delta b||_2}{||b||_2},$$

where $\mathrm{cond}(A)$ is the condition number of the matrix $A \in \mathbb{R}^{n\times n}$.

3. Consider the function $g(x) = 2^{-x}$ on the interval $[0.5, 1]$.

(a) Prove that g has a unique fixed point p on this interval.
(b) Prove that applying the fixed point iteration to this problem converges.
(c) Estimate the number of iterations needed to achieve an accuracy of 10^{-5} when applying the fixed point iteration for approximating p.

Justify your answers.

4. Suppose D_h is the following difference operator that approximates $\frac{d^2u}{dx^2}$ using grid spacing $h > 0$:

$$D_h u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}.$$

(a) Devise a combination of D_h and D_{2h} that yields a 4th order approximation to $\frac{d^2u}{dx^2}$.
(b) Give a derivation of the leading order term in the local truncation error for the difference approximation you obtained in Part (a).

Justify your answers.

5. Consider the initial value problem:

$$\begin{aligned} y'(t) &= f(t, y(t)), \quad \text{for} \quad t \in [0, T], \\ y(0) &= c, \end{aligned}$$

and assume that $y(t) \in C^3[0, T]$. Recall that the midpoint method is given by:

$$y_{j+1} = y_j + h\, f\left(t_j + \frac{h}{2},\, y_j + \frac{h}{2} f(t_j, y_j)\right).$$

Show that the local truncation error for the midpoint method:

$$\tau_{j+1}(h) = \frac{y(t_{j+1}) - y(t_j)}{h} - f\left(t_j + \frac{h}{2},\, y(t_j) + \frac{h}{2} f(t_j, y(t_j))\right)$$

is $\mathcal{O}(h^2)$.

6. Consider the partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \mu \frac{\partial^2 u}{\partial x^2}, \quad \text{where } \mu > 0 \text{ is constant,}$$

to be solved for $x \in [0, 1]$ and $t > 0$, with periodic boundary conditions and initial data $u(x, 0) = u_0(x)$. The initial data is assumed to be positive and smooth.

(a) Devise a finite difference approximation which converges as Δt and Δx go to zero for all $\mu > 0$ and remains convergent as $\mu \to 0^+$. You may or may not want to use the fact that $u_0(x)$ is positive.

(b) What is the bound on Δt in terms of Δx and μ?

Justify your answers.

7. Obtain a convergent finite difference approximation for the equation

$$u_{tt} = u_{xx} + u_{yy} + u_x$$

to be solved for $x \in [0, 1]$, $y \in [0, 1]$, and $t > 0$, with periodic boundary condition in both x and y and smooth initial data:

$$\begin{aligned} u(x, y, 0) &= v(x, y) \\ u_t(x, y, 0) &= w(x, y). \end{aligned}$$

Justify your answers.

8. Consider the following elliptic boundary value problem in two-dimensions:

$$\begin{aligned} -\Delta u + u = f(x,y), &\quad \text{in } \Omega = [0,1]\times[0,1],\\ u(0,y) = 0, &\quad \text{for } \ y\in[0,1],\\ u(1,y) = 0, &\quad \text{for } \ y\in[0,1],\\ u_y(x,0) = 0, &\quad \text{for } \ x\in[0,1],\\ u_y(x,1) = 0, &\quad \text{for } \ x\in[0,1]. \end{aligned}$$

(a) Give a weak variational formulation of the problem.
(b) Analyze the existence and uniqueness of solutions to this problem, assuming $f \in L^2(\Omega)$.
(c) Develop and describe the piecewise linear Galerkin finite element approximation of the problem. Discuss the form and properties of the stiffness matrix and the existence and uniqueness of the solution to the corresponding linear algebraic problem.

Justify your answers.