Qualifying Exam, Spring 2024

Optimization / Numerical Linear Algebra (ONLA)

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM. PLEASE USE BLANK PAGES AT END FOR ADDITIONAL SPACE.

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1. (10 points) Consider

$$\begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Prove whether Gauss-Seidel iteration converges.

2. (10 points) Consider the following iterative method for solving $Ax = b$:

$$
\begin{aligned}
&r_0 = b - Ax_0 \\
&p_0 = r_0 \\
&\text{For } n = 0, 1, \ldots \\
&\qquad \text{Compute } \alpha_n \\
&\qquad x_{n+1} = x_n + \alpha_n p_n \\
&\qquad r_{n+1} = r_n - \alpha_n A p_n \\
&\qquad \text{Compute } p_{n+1}
\end{aligned}
$$

Derive a way to compute α_n (in the fourth line above) such that $\|b - Ax_{n+1}\|_2$ is minimized.

3. (10 points) Prove for Krylov subspace $\mathcal{K}_n(A, v)$:

   a) $\mathcal{K}_n(A, v) = \mathcal{K}_n(\alpha A, \beta v)$ where $\alpha \neq 0$, $\beta \neq 0$.

   b) $\mathcal{K}_n(A, v) = \mathcal{K}_n(A - \mu I, v), \forall \mu \in \mathbb{R}$.

   c) $\mathcal{K}_n(B^{-1}AB, B^{-1}v) = B^{-1}\mathcal{K}_n(A, v), \forall B$ that's invertible.

   d) $\mathcal{K}_{n+1}(A, v) = \text{span}(v) + A\mathcal{K}_n(A, v), \forall n \geq 1$.

   e) If $v \notin \text{Range}(A)$, then $A\mathcal{K}_n(A, v) \not\subseteq \mathcal{K}_{n+1}(A, v), \forall n \geq 1$.

4. (10 points) Prove or disprove whether each of the following is backward stable, when performed in floating point arithmetic on a machine that satisfies the fundamental axiom of floating point arithmetic (i.e. that floating point operations $\circledast$ satisfy $x \circledast y = (x * y)(1+\varepsilon)$ for some $|\varepsilon| \leq \epsilon_{\text{machine}}$, where $*$ represents addition, subtraction, multiplication or division).

   a) Outer product computations xy^T for real vectors x and y.

   b) Unitary matrix multiplication. I.e., let Q be a unitary $m \times m$ real matrix and define the problem f by $f(A) := QA$ for $m \times m$ real matrices A. Suppose this is carried out by the floating point algorithm $\hat{f}(A)$ which computes the product QA by floating point inner products. Prove or disprove that $\hat{f}$ is backward stable.

5. (10 points) Given an $m \times n$ real matrix A and real vector $b \in \mathbb{R}^m$, consider the least-squares problem in which we search for the least-squares solution $x_{LS} \in \mathbb{R}^n$ that minimizes $f(x) = \|Ax - b\|_2^2$. Prove that $x_{LS} = A^\dagger b$ is the least-squares solution that has the smallest L2-norm (i.e. show that it is a minimizer, and out of all minimizers, $\|A^\dagger b\|_2$ is the smallest). The notation $A^\dagger$ denotes the (Moore-Penrose) pseudo-inverse of A.

6. (10 points) Prove Gershgorin's theorem: Let A be a square matrix with entries a_{ij} and denote by D_i the disc centered at a_{ii} with radius $r_i = \sum_{i \neq j} |a_{ij}|$. Then every eigenvalue of A lies within at least one disc D_i.

7. (10 points) Consider the problem to extremize (over $\mathbb{R}^2$),

$$x_1^2 + x_2^2 \quad \text{subject to} \quad x_1^2 + x_2^3 \leq 1.$$

a) Write down the KKT conditions for this problem and find all points that satisfy them.

b) Determine whether or not these points satisfy the second order necessary conditions for being local maximizers or minimizers.

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8. (10 points) Consider the problem (over $\mathbb{R}^2$),

$$\begin{aligned} \text{minimize} \quad & x_1^4 - 2x_2^2 - x_2 \\ \text{subject to} \quad & x_1^2 + x_2^2 + x_2 \leq 0 \end{aligned}$$

a) Write a dual problem and solve it.

b) Using duality, find a solution for the original problem.

9. (10 points) Consider a quadratic function $f(x) = \frac{1}{2}x^T Qx - b^T x$, where Q is an $n \times n$ symmetric positive definite matrix. Consider the steepest descent iteration for minimizing this function, which is defined by

$$x_{k+1} = x_k - \alpha_k g_k, \quad \alpha_k = \operatorname{argmin}_{\alpha \geq 0} f(x_k - \alpha g_k), \quad g_k = \nabla f(x_k).$$

Note $x \in \mathbb{R}^n$ and the notation x_k denotes the iterate in the kth iteration, which is also a vector in $\mathbb{R}^n$.

a) Show that $\alpha_k = \frac{g_k^T g_k}{g_k^T Q g_k}$.

b) Denoting the minimizer of f by x^* and using the definition $\|x\|_Q^2 = x^T Qx$, show that

$$\|x_{k+1} - x^*\|_Q^2 = \left\{1 - \frac{(g_k^T g_k)^2}{(g_k^T Q g_k)(g_k^T Q^{-1} g_k)}\right\} \|x_k - x^*\|_Q^2.$$

c) Denoting the eigenvalues of Q as $0 < \lambda_1 \leq \cdots \leq \lambda_n$, show that for any vector v one has

$$\frac{(v^T v)^2}{(v^T Q v)(v^T Q^{-1} v)} \geq \frac{4\lambda_1 \lambda_n}{(\lambda_1 + \lambda_n)^2},$$

and use this to conclude that $\|x_{k+1} - x^*\|_Q^2 \leq \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\right)^2 \|x_k - x^*\|_Q^2$.