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Development of a Fourier–Stieltjes transform using an induced representation on locally compact groups

Yao Ihébami Akakpo[1,2,3] · **Mahouton Norbert Hounkonnou**[3] · **Koffi Enakoutsa**[4] · **Kofi V. S. Assiamoua**[3,5]

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Abstract

In our research, we broaden the scope of Fourier–Stieltjes transforms to encompass locally compact groups, denoted as G. We achieve this extension by leveraging the induced representation from a closed subgroup K. From this, we deduce the Fourier transform $\hat{f}$ of a Haar-integrable function f defined on G. Specifically, we express $\hat{f}$ as the Fourier–Stieltjes transform $\hat{\mu}$ of the measure $\mu = f\lambda$, where λ denotes the Haar measure of G. Our work is significant because when applied to Lie groups with compact subgroups K, our Fourier–Stieltjes transform $\hat{m}$ exhibits more nuanced characteristics compared to the traditionally defined one via the Gel'fand transform, which is standard in the context of Lie groups. We rigorously substantiate this observation. One of the principal challenges we confront is the construction of the "trigonometric functions", which serve as the foundation for building the Fourier transform.

Keywords Fourier–Stieltjes · Induced representation · Locally compact group · Vector measure · Banach space

✉ Mahouton Norbert Hounkonnou
norbert.hounkonnou@cipma.uac.bj; hounkonnou@yahoo.fr

✉ Kofi V. S. Assiamoua
kofassiam@yahoo.fr

Yao Ihébami Akakpo
akjeremiji@yahoo.fr; akakpo.yao_ihebami@courrier.uqam.ca; yao.ihebami.akakpo@cipma.uac.bj

Koffi Enakoutsa
koffi@math.ucla.edu

1 1300 Rue Ouimet App7, Montréal, QC H4L 2L3, Canada

2 ICMPA-UNESCO Chair (UAC), 072 BP 50, Cotonou, Republic of Benin

3 International Chair in Mathematical Physics and Applications (ICMPA-UNESCO Chair), University of Abomey-Calavi, 072 BP 50, Cotonou, Republic of Benin

4 Department of Mathematics, University of California Los Angeles, 520 Portola Plaza, Los Angeles, CA 90095, USA

5 Department of Mathematics, Faculté Des Sciences, University of Lomé, BP 1515, Lomé, Togo

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1 Introduction

The vector measures generalizing scalar measures attracted a great interest in recent decades due to their numerous applications in functional analysis, control systems, signal analysis, quantum information, quantum theories, and many other domains of applications. For more details on vector measure theory, see for instance, [4, 9] and [1] for some applications on compact groups. Also, Clarkson [6] used theoretical ideas on vector measures to prove that many Banach spaces do not admit equivalent uniformly convex norms. In the same vein, Gel'fand [14] proved that $L_1[0, 1]$ is not isomorphic to a dual of a Banach space. Lyapunov [20] showed that the range of a (non-atomic) vector measure is closed and convex. Lyapunov [20]' s work occupies a prominent place in modern mathematics since it lies at the intersection of the theory of convex sets and measure theory. The Lyapunov convexity theorem became the starting point of numerous studies in the framework of mathematical analysis as well as in the realm of geometric research into the convex sets that are ranges of non-atomic vector measures [18]. In addition, Bartle [2], Dinculeanu, Kluvánek [9], Dunford and Schwartz [10], and Lindenstrauss and Pelczyńki [19] gave many seminal results on vector measure. For instance, Diestel and Uhl Jr [8] provided a comprehensive survey on vector measures. Applications of vector measures were discussed in the work by Kluvánek [17]. Fernández and Faranjo [11] studied the Rybakov's theorem for vector measures in Fréchet spaces. Curbera and Ricker [7] wrote a survey on vector measures, integration, and applications. More information on vector measures can also be found in [7, 8].

To focus on our interest, let G be a locally compact group, m a vector measure on G into a Banach algebra $\mathcal{A}$, λ a left or right Haar measure on G, and $f \in L_1(G, \lambda)$.

If the group G is abelian, the Fourier–Stieltjes transform of m is given by the relation

$$\hat{m}(\chi) = \int_G \overline{\langle \chi, t \rangle} dm(t), \tag{1}$$

while the Fourier transform of f is given by

$$\hat{f}(\chi) = \int_G \overline{\langle \chi, t \rangle} f(t) d\lambda(t), \tag{2}$$

where χ denotes a character of G.

If G is compact and $\mathcal{A} = \mathbb{C}$, then the Fourier–Stieltjes transform of m is a family of endomorphisms $(\hat{m}(\sigma))_{\sigma \in \Sigma}$ given by

$$\left\langle \hat{m}(\sigma)\xi, \eta \right\rangle = \int_G \left\langle \overline{U}_t^{\sigma} \xi, \eta \right\rangle dm(t), \tag{3}$$

and the associated Fourier transform is provided by the relation

$$\left\langle \hat{f}(\sigma)\xi, \eta \right\rangle = \int_G \left\langle \overline{U}_t^{\sigma} \xi, \eta \right\rangle f(t) d\lambda(t), \tag{4}$$

where U^{σ} denotes a unitary representation of the group G.

If G is compact and $\mathcal{A}$ is any arbitrary Banach algebra, Assiamoua [1] defined a Fourier–Stieltjes transform of a bounded vector measure m on G as a family $(\hat{m}(\sigma))_{\sigma \in \Sigma}$ of sesquilinear mappings of $H_\sigma \times H_\sigma$ with values in $\mathcal{A}$ given by the relation,

$$\hat{m}(\sigma)(\xi, \eta) = \int_G \left\langle \overline{U}_t^{\sigma} \xi, \eta \right\rangle dm(t), \tag{5}$$

and the Fourier transform of a function $f \in L_1(G)$ as a family of continuous endomorphisms $(\hat{f}(\sigma))_{\sigma\in\Sigma}$ of sesquilinear applications of $H_\sigma \times H_\sigma$ with value in $\mathcal{A}$, given by the relation

$$\hat{f}(\sigma)(\xi,\eta) = \int_G \left\langle \overline{U}_t^\sigma \xi, \eta \right\rangle f(t) d\lambda(t). \tag{6}$$

In the continuation of previous investigations by [1], the present work addresses a construction of the Fourier–Stieltjes transform on locally compact groups from a group representation induced by a representation of a compact subgroup. For this purpose, we consider a locally compact group G, K a compact subgroup of G, μ a G-invariant measure on the left coset space G/K, and L^σ a unitary representation of K into a separable Hilbert space H_σ. Then, we define the Fourier–Stieltjes transform of a bounded vector measure on the locally compact group G using the representation U^{L^σ} of G induced by L^σ. In this context one usually uses Gel'fand transform to define the Fourier transform. Given a locally compact group G with Haar measure dx, a unitary commutative Banach algebra A, and $X(A)$ its spectrum, the Gel'fand transform of x, $x \in A$, is the function $\mathcal{G}_x : X(A) \longrightarrow \mathbb{C}$ such that $\mathcal{G}_x(\chi) = \chi(x)$. The mapping $x \longmapsto \mathcal{G}_x : A \longrightarrow \mathbb{C}^{X(A)}$ is called the transformation of Gel'fand associated with A. The spherical Fourier transform is the Gel'fand transform associated with $L_1(G)^\natural$, the space of integrable, bi-invariant functions by a compact subgroup K of G on G. In this case, for $f \in L_1(G)^\natural$, $\mathcal{G}_x$ is denoted $\mathcal{F}f$ or $\hat{f}$ and is defined by

$$\hat{f}(\chi) = \int_G f(x)\chi(x^{-1})dx. \tag{7}$$

Our method of construction of Fourier transform has several advantages over the Gel'fand transform. First, our transform is injective while the Gel'fand transform is not always injective. Second, the Gel'fand transform is limited to spherical functions only while our Fourier transform exists for the whole $L_1(G, A)$, $1 \leq p < \infty$. Finally, our method is constructive while in Gel'fand transformation the Fourier transform is obtained by induction.

The paper is organized as follows. In Sect. 1, we recall the definition of a vector measure and the unit representation of a group which are useful in the next sections. In Sect. 2, we provide the proof of the Shur's orthogonality property in connection with induced representation and we define the Fourier–Stieltjes transform of a vector measure and the Fourier transform of a function in $L_1(G, \mathcal{A})$. Also, we report several properties we discovered for our newly developed Fourier–Stieltjes transform in Sect. 2.

2 Preliminaries

In this section, for the clarity of the development, we briefly recall useful known main definitions and results, and set our notations. We consider a locally compact space G, the Banach spaces $\mathcal{A}$ and $\mathcal{F}$ over the field $\mathbb{K}$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and denote by $\mathcal{K}(G, \mathcal{A})$ the vector space of all continuous functions $f : G \longrightarrow \mathcal{A}$ with a compact support, and by $\mathfrak{C}(G, \mathcal{A})$ the space of continuous functions $f : G \longrightarrow \mathcal{A}$.

For simplification, we write $\mathcal{K}(G)$ instead of $\mathcal{K}(G, \mathbb{R})$ or $\mathcal{K}(G, \mathbb{C})$. For each subset K of G, let denote by $\mathcal{K}_K(G, \mathcal{A})$ the space of functions with support contained in K. $\mathcal{K}_K(G, \mathcal{A})$ is a subspace of $\mathcal{K}(G, \mathcal{A})$.

Definition 2.1 For every function $f \in \mathcal{K}(G, \mathcal{A})$, we define

$$\|f\| := \sup_{t\in G} \|f(t)\|_{\mathcal{A}}.$$

The mapping $f \mapsto \|f\|$ is a norm on each space $\mathcal{K}_K(G, \mathcal{A})$; it defines the topology of uniform convergence on G over $\mathcal{K}(G)$.

Definition 2.2 On $\mathcal{K}(G, \mathcal{A})$, the topology of the compact convergence is the locally convex topology defined by the family of seminorms

$$\|f\|_K = \sup_{t \in K} \|f(t)\|_{\mathcal{A}},$$

where K takes the elements in the set of compact subsets of G.

Proposition 2.3 *The space $\mathcal{K}(G, \mathcal{A})$ is dense in the space $\mathfrak{C}(G, \mathcal{A})$ for the topology of the compact convergence* [9].

Definition 2.4 A vector measure on G with respect to two spaces $\mathcal{A}$ and $\mathcal{F}$, or an $(\mathcal{A}, \mathcal{F})$-measure on G, is any linear mapping $m : \mathcal{K}(G, \mathcal{A}) \longrightarrow \mathcal{F}$ having the property that, for each compact set $K \subset G$, the restriction m to the subspace $\mathcal{K}_K(G, \mathcal{A})$ is continuous for the topology of uniform convergence, i.e. for each compact set, there exists a number $a_K > 0$ such that

$$\|m(f)\| \leq a_K \sup \{\|f(t)\|_{\mathcal{A}}, \quad t \in K\}.$$

The value $m(f)$ of m for a function $f \in \mathcal{K}(G, \mathcal{A})$ is called the integral of f with respect to m also denoted by $\int_G f dm$ or $\int_G f(t)dm(t)$. A vector measure is said to be dominated if there exists a positive measure μ such that

$$\left\| \int_G f(t)dm(t) \right\| \leq \int_G |f(t)| d\mu(t), \qquad f \in \mathcal{K}(G).$$

If m is dominated, then there exists a smallest positive measure $|m|$, called the modulus or the variation of m, that dominates it. A positive measure is said to be bounded if it is continuous in the uniform norm topology of $\mathcal{K}(G)$. A vector measure is said to be bounded if it is dominated by a bounded positive measure. If m is bounded, then $|m|$ is also bounded.

Denoting by $M_1(G, \mathcal{A})$ the Banach algebra of bounded vector measures on G, the mapping

$$m \mapsto \|m\| = \int_G \chi_G d|m| \tag{8}$$

is a norm on $M_1(G, \mathcal{A})$, where χ_G represents the characteristic function of G.

In the sequel, K will denote a compact subgroup of G, ν and λ, the left Haar measures on K and G, respectively.

Definition 2.5 Let μ be a Radon measure on G/K, the homogenous space of left K-cosets and g an element of G. Define μ_g by $\mu_g(E) = \mu(gE)$ for the Borel subsets E of G/K. The measure μ is called $G-$invariant measure if $\mu_g = \mu$, for $g \in G$, see [12, 13] for more details.

Throughout the paper, μ will denote the $G-$ invariant measure on G/K. If ν denotes a Haar measure in K, we have the following theorem:

Theorem 2.6 *For any $f \in \mathcal{K}(G)$, we have* [12, 15, 16]*:*

$$\int_G f(g)d\lambda(g) = \int_{G/K} d\mu(\dot{g}) \int_K f(gk)d\nu(k). \tag{9}$$

Theorem (2.6) also extends to every $f \in L_1(G, \lambda, A)$, see [5].

Definition 2.7 A unit representation of G is a homomorphism L from G into the group $U(H)$ of the invertible unitary linear operators on some nonzero Hilbert space H, which is continuous with respect to the strong operator topology satisfying for $g_1, g_2 \in G$,

$$L_{g_1 g_2} = L_{g_1} L_{g_2} \quad \text{and} \quad L_1 = Id_H.$$

H is called the representation space of L, and its dimension is called the dimension or degree of L.

Suppose $\mathcal{M}$ is a closed subspace of H. $\mathcal{M}$ is called an invariant subspace for L if $L_g \mathcal{M} \subset \mathcal{M}\ \forall g \in G$. If $\mathcal{M}$ is invariant and $\mathcal{M} \neq \{0\}$, then $L^{\mathcal{M}}$ such that

$$L_g^{\mathcal{M}} = L_g|_{\mathcal{M}}$$

defines a representation of G on $\mathcal{M}$, called a subrepresentation of L. If L admits an invariant subspace that is nontrivial (i.e. $\neq \{0\}$ or H) then L is called reducible, otherwise L is irreducible. If G is compact and L irreducible then the dimension of L is finite.

Definition 2.8 Two unit irreducible representations L and V into H and N, respectively, are said to be equivalent if there is an isomorphism $T : H \longrightarrow N$ such that, $\forall t \in G$,

$$T \circ L_t = V_t \circ T.$$

Consider now the subgroup K, Σ the coset space of the class of irreducible representations of K (called the dual object of K), $\sigma \in \Sigma$, L^σ a representative of σ, H_σ a representation space of L^σ, and $d\sigma$ its dimension.

Theorem 2.9 *Let $(L^\sigma_{ij})_{1\le i,j\le d_\sigma}$ be the matrix of L^σ in an orthonormal basis $(\xi_i)_{i=1}^{d\sigma}$ of H_σ. Then, (see* [3, 12, 16, 21]*),*

$$\int_K L^\sigma_{ij}(t)\overline{L^\sigma_{lm}}(t)d\nu(t) = \frac{\delta_{il}\delta_{jm}}{d_\sigma} \tag{10}$$

and

$$\int_K L^\sigma_{ij}(t)\overline{L^\tau_{lm}}(t)d\nu(t) = 0 \text{ if } \sigma \neq \tau. \tag{11}$$

Let $q : G \longrightarrow G/K$ be the canonical quotient map of G into G/K and suppose H_σ separable. Denote by $H_0^{L^\sigma}$ the set

$$H_0^{L^\sigma} = \left\{u \in \mathfrak{C}(G, H_\sigma) \ : \ q(\text{Supp(u)}) \text{ is compact and } u(gk) = L^\sigma_{k^{-1}} u(g)\right\}. \tag{12}$$

Proposition 2.10 *If $\eta : G \longrightarrow H_\sigma$ is continuous with compact support, then the function u_η such that*

$$u_\eta(g) = \int_K L^\sigma_k \eta(gk)d\nu(k) \tag{13}$$

belongs to $H_0^{L^\sigma}$, and is uniformly continuous on G. Moreover, every element of $H_0^{L^\sigma}$ is of the form u_η, see [12] *for more details.*

Proposition 2.11 *The mapping:*

$$(u, v) \longmapsto \langle u, v\rangle = \int_{G/K} \langle u(g), v(g)\rangle_{H_\sigma}\, d\mu(\dot{g}). \tag{14}$$

on $H_0^{L^\sigma} \times H_0^{L^\sigma}$ *is an inner product on* $H_0^{L^\sigma}$.

G acts on $H_0^{L^\sigma}$ by left translation, $u \longmapsto L_t u$, so we obtain a unitary representation of G with respect to this inner product on $H_0^{L^\sigma}$. The inner product is preserved by left translations, since μ is invariant. Hence, if we denote by H^{L^σ} the Hilbert space completion of $H_0^{L^\sigma}$, the translation operators L_t extend to unitary operators on H^{L^σ}. Then the map $t \longmapsto L_t u$ is continuous from G to H^{L^σ} for each $u \in H_0^{L^\sigma}$, and then, since the operators L_t are uniformly bounded, they are strongly continuous on H^{L^σ}. Hence, they define a unitary representation of G, called the representation induced by L^σ, denoted by U^{L^σ}:

$$U_t^{L^\sigma} u(g) = L_t u(g) = u(t^{-1}g).$$

The representation space is denoted H^{L^σ}.

Remark 2.12
- The representations of G induced from K are generally infinite-dimensional unless G/K is a finite set.
- If the induced representation U^{L^σ} is irreducible then L^σ is irreducible [3]. The converse of this statement is false.
- $U^{L^\sigma} \in U(H^{L\sigma})$, where $U(H^{L\sigma})$ denotes the group of invertible unitary linear operators on $H^{L\sigma}$.

Lebesgue Spaces Let λ be a positive measure on the locally compact group G, and $\mathcal{A}$ Banach algebra. For a function $f : G \longrightarrow \mathcal{A}$, we define

$$N_p(f) = \left(\int_G \|f(t)\|_{\mathcal{A}}^p\, d\lambda(t)\right)^{\frac{1}{p}} \tag{15}$$

for $1 \leq p < \infty$, where $\int_G$ denotes the upper integral [4], and

$$N_\infty(f) = \inf\{\alpha : \|f(t)\|_{\mathcal{A}} \leq \alpha, \quad \lambda - \text{almost everywhere}\}. \tag{16}$$

We denote by $\mathcal{L}_p(G, \lambda, \mathcal{A})$ the set of all λ-measurable functions $f : G \longrightarrow \mathcal{A}$ such that $N_p(f) < \infty$, $1 \leq p \leq \infty$. The mapping $f \longmapsto N_p(f)$ is seminorm in $\mathcal{L}_p(G, \lambda, \mathcal{A})$.

Definition 2.13 $L_p(G, \lambda, \mathcal{A})$ will denote the set of equivalence classes $[f]$ of functions $f \in \mathcal{L}_p(G, \lambda, \mathcal{A})$ where $g \in [f]$ means that $f - g$ is null $\lambda-$ almost everywhere

In the following, f will be used instead of $[f]$ to denote an element in $L_p(G, \lambda, \mathcal{A})$

Theorem 2.14 $\|.\|_p = N_p$ *is a norm in* $L_p(G, \lambda, \mathcal{A})$. *With this norm,* $L_p(G, \lambda, \mathcal{A})$ *becomes a Banach space.*

Proposition 2.15 $L_p(G, \lambda, \mathcal{A}) \cap L_q(G, \lambda, \mathcal{A})$ *is dense in* $L_p(G, \lambda, \mathcal{A})$ *and in* $L_q(G, \lambda, \mathcal{A})$.

Remark 2.16 If m is a dominated vector measure of G with respect to $\mathcal{A}$ and a Banach space $\mathcal{F}$, then $\mathcal{L}_p(G, m, \mathcal{A})$ is by convention the space $\mathcal{L}_p(G, |m|, \mathcal{A})$.

Remark 2.17 Assuming that λ is a Haar measure of G, for any $f \in L_1(G, \lambda, \mathcal{A})$ $f\lambda$ is a vector measure absolutely continue with respect to λ and for any $g \in \mathcal{K}(G, \mathcal{A})$

$$f\lambda(g) = \int_G f(t)g(t)d\lambda(t). \tag{17}$$

3 Main results

Assume that K is compact and H_σ is both finite and separable, then H^{L^σ} also exhibits separability. Moreover, each of these spaces possesses a Hilbertian basis constructed through the Gram-Schmidt process. In our investigation, we assume the selection of K such that both L^σ and U^{L^σ} are irreducible.

3.1 Schur orthogonaltity relations

Let $(\theta_i)_{i=1}^{\infty}$ be an orthonormal basis of H^{L^σ} and let $(\xi_i)_{i=1}^{d\sigma}$ be an orthonormal basis of H_σ. We define

$$u_{ij}^{L^\sigma}(t) := \left\langle U_t^{L^\sigma}\theta_j, \theta_i \right\rangle_{H^{L^\sigma}} = \int_{G/K} \left\langle \theta_j(t^{-1}g), \theta_i(g) \right\rangle d\mu(\dot{g}) \tag{18}$$

and

$$L_{ij}^\sigma(k) := \left\langle L_k^\sigma \xi_j, \xi_i \right\rangle_{H_\sigma}.$$

As a result, there is a family of mappings $(\alpha_{is})_{s=1}^{d\sigma}$ of G into $\mathbb{K}$ $with$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) such that

$$\theta_i(g) = \sum_{s=1}^{d\sigma} \alpha_{is}(g)\xi_s.$$

We have:

$$\delta_{ij} = \left\langle \theta_j, \theta_i \right\rangle_{H^{L_\sigma}} = \int_{G/K} \left\langle \theta_j(g), \theta_i(g) \right\rangle_{H_\sigma} d\mu(\dot{g}) = \sum_{s=1}^{d\sigma} \int_{G/K} \alpha_{js}(g)\overline{\alpha}_{is}(g) d\mu(\dot{g}).$$

Proposition 3.1 $\forall \sigma \in \Sigma$ *and* $\forall i, j \in \{1, 2, ...\}$, $u_{ij}^{L^\sigma} \in \mathcal{K}(G)$.

Proof The continuity of $u_{ij}^{L^\sigma}$ results from their construction.
According to Eq. (13), for all $u \in H^{L^\sigma}$, there is $\eta \in \mathcal{K}(G, H_\sigma)$ such that $u_\eta(g) = \int_K L_k^\sigma \eta(gk) d\nu(k)$. It follows that $q(supp(u)) \subset q(supp(\eta))$.

Let A be the support of η. For every $g \in G$ and $k \in K$, $gk \in A \Longrightarrow g \in Ak^{-1}$. $AK = \bigcup_{k \in K} Ak^{-1}$ is compact because A and K are compact. Furthermore, $\forall g \notin AK$, $u(g) = 0$ then $supp(u) \subset AK$. Since $supp(u)$ is a closed subset of AK then it is compact.

Now let B and C be $supp(\theta_j)$ and $supp(\theta_i)$, respectively. Assume $D = \left\{t \in G : g \in C \Longrightarrow t^{-1}g \in B\right\}$. According to Eq. (18), $\left\langle \theta_j(t^{-1}g), \theta_i(g) \right\rangle = 0$ if $t \notin D$, and then $u_{ij}^{L^\sigma}(t) = 0$, then $supp(u_{ij}^{L^\sigma}) \subset D$. In addition,

$$t^{-1}g \in B \text{ with } g \in C \Longrightarrow t \in CB^{-1}.$$

Then $supp(u_{ij}^{L^\sigma}) \subset D \subset CB^{-1}$. Since CB^{-1} is compact, $supp(u_{ij}^{L^\sigma})$ is also compact. □

Therefore, we have

Corollary 3.2

$$\int_G \left| u_{ij}^{L^\sigma}(t)\overline{u}_{lm}^{L^\sigma}(t) \right| d\lambda(t) < \infty.$$

We have $K\backslash G = G/K$ and therefore

Lemma 3.3 *$\forall u \in H^{L^\sigma}$, $u(kt) = L_k u(t)$ and $u(tk) = L_{k^{-1}}^\sigma u(t)$.*

Proof There is $\eta \in \mathfrak{C}(G, H_\sigma)$ such that

$$\begin{aligned} u(t) &= \int_K L^\sigma(\xi)\eta(t\xi)d\nu(\xi). \quad [12] \\ u(tk) &= \int_K L^\sigma(\xi)\eta(tk\xi)d\nu(\xi), \quad k \in K \\ &= \int_K L^\sigma(k^{-1}\xi)\eta(t\xi)d\nu(\xi) \\ &= L_{k^{-1}}^\sigma \int_K L^\sigma(\xi)\eta(t\xi)d\nu(\xi) \\ &= L_{k^{-1}}^\sigma u(t). \end{aligned}$$

There exists $\beta \in \mathfrak{C}(G, H_\sigma)$ such that

$$\begin{aligned} u(t) &= \int_K L^\sigma(\xi^{-1})\beta(\xi t)d\nu(\xi). \quad [3] \\ u(kt) &= \int_K L^\sigma(\xi^{-1})\beta(\xi kt)d\nu(\xi) \\ &= \int_K L^\sigma(k\xi^{-1})\beta(\xi t)d\nu(\xi) \\ &= L_k^\sigma \int_K L^\sigma(\xi^{-1})\beta(\xi t)d\nu(\xi) \\ &= L_k^\sigma u(t). \end{aligned}$$

□

The following theorem shows the Schur orthogonality relation for the the case of the representation U^{L^σ} of G induced by the unitary irreducible representation L^σ of K.

Theorem 3.4 *We have*

$$\int_G u_{ij}^{L^\sigma}(t)\overline{u}_{lm}^{L^\sigma}(t)d\lambda(t) = \frac{c_{ijlm}}{d_\sigma} \tag{19}$$

where

$$\begin{aligned} c_{ijlm} &:= \int_{G/K} d\mu(\dot{t}) \sum_{r,s=1}^{d_\sigma} \int_{G/K} \alpha_{js}(t^{-1}g)\overline{\alpha}_{ir}(g)d\mu(\dot{g}) \int_{G/K} \alpha_{ms}(t^{-1}h)\overline{\alpha}_{lr}(h)d\mu(\dot{h}) \\ &= \sum_{r,s=1}^{d_\sigma} \int_{(G/K)^3} \alpha_{js}(t^{-1}g)\overline{\alpha}_{ir}(g)\alpha_{ms}(t^{-1}h)\overline{\alpha}_{lr}(h)d\mu(\dot{g})d\mu(\dot{h})d\mu(\dot{t}) \end{aligned}$$

and

$$\int_G u_{ij}^{L^\sigma}(t)\overline{u}_{lm}^{L^\tau}(t)d\lambda(t) = 0 \text{ if } \sigma \neq \tau. \tag{20}$$

Proof By a straightforward computation, we obtain:

$$\begin{aligned}\int_{G/K}\left\langle\theta_j(gk),\theta_i(g)\right\rangle_{H_\sigma}d\mu(\dot g) &= \int_{G/K}\left\langle L^\sigma(k^{-1})\theta_j(g),\theta_i(g)\right\rangle_{H_\sigma}d\mu(\dot g)\\ &= \sum_{s,r=1}^{d\sigma}\int_{G/K}\alpha_{js}(g)\overline{\alpha}_{ir}(g)\left\langle L^\sigma(k^{-1})\xi_s,\xi_r\right\rangle_{H_\sigma}d\mu(\dot g)\\ &= \sum_{r,s=1}^{d_\sigma}L^\sigma_{rs}(k^{-1})\int_{G/K}\alpha_{js}(g)\overline{\alpha}_{ir}(g)d\mu(\dot g).\end{aligned}$$

Besides, we get:

$$\begin{aligned}u^{L^\sigma}_{ij}(tk) &= \int_{G/K}\left\langle\theta_j(k^{-1}t^{-1}g),\theta_i(g)\right\rangle d\mu(\dot g)\\ &= \int_{G/K}\left\langle L^\sigma(k^{-1})\theta_j(t^{-1}g),\theta_i(g)\right\rangle d\mu(\dot g)\\ &= \int_{G/K}\left\langle L^\sigma(k^{-1})\theta_j(t^{-1}g),\theta_i(g)\right\rangle d\mu(\dot g)\\ &= \sum_{r,s=1}^{d_\sigma}L^\sigma_{rs}(k^{-1})\int_{G/K}\alpha_{js}(t^{-1}g)\overline{\alpha}_{ir}(g)d\mu(\dot g)\end{aligned}$$

and

$$\begin{aligned}\int_G u^{L^\sigma}_{ij}(t)\overline{u}^{L^\sigma}_{lm}(t)d\lambda(t) &= \int_{G/K}\int_K u^{L^\sigma}_{ij}(tk)\overline{u}^{L^\sigma}_{lm}(tk)d\nu(k)d\mu(\dot t)\\ &= \int_{G/K}d\mu(\dot t)\int_K\Big(\sum_{r,s=1}L^\sigma_{rs}(k^{-1})\int_{G/K}\alpha_{js}(t^{-1}g)\overline{\alpha_{ir}}(g)d\mu(\dot g)\\ &\quad\times\sum_{p,q=1}L^\sigma_{pq}(k^{-1})\int_{G/K}\alpha_{mq}(t^{-1}h)\overline{\alpha_{lp}}(h)d\mu(\dot h)\Big)d\nu(k)\\ &= \int_{G/K}d\mu(\dot t)\sum_{r,s,p,q=1}\int_K\Big(L^\sigma_{rs}(k^{-1})\int_{G/K}\alpha_{js}(t^{-1}g)\overline{\alpha_{ir}}(g)d\mu(\dot g)\\ &\quad\times L^\sigma_{pq}(k^{-1})\int_{G/K}\alpha_{mq}(t^{-1}h)\overline{\alpha_{lp}}(h)d\mu(\dot h)\Big)d\nu(k).\end{aligned}$$

Thus,

$$\begin{aligned}\int_G u^{L^\sigma}_{ij}(t)\overline{u}^{L^\sigma}_{lm}(t)d\lambda(t) &= \int_{G/K}d\mu(\dot t)\sum_{r,s,p,q=1}\int_K\big(L^\sigma_{rs}(k^{-1})L^\sigma_{pq}(k^{-1})d\nu(k)\\ &\quad\int_{G/K}\alpha_{js}(t^{-1}g)\overline{\alpha_{ir}}(g)d\mu(\dot g)\\ &\quad\times\int_{G/K}\alpha_{mq}(t^{-1}h)\overline{\alpha_{lp}}(h)d\mu(\dot h)\\ &= \int_{G/K}d\mu(\dot t)\sum_{r,s,p,q=1}\frac{\delta_{rp}\delta_{sq}}{d_\sigma}\int_{G/K}\alpha_{js}(t^{-1}g)\overline{\alpha_{ir}}(g)d\mu(\dot g)\\ &\quad\times\int_{G/K}\alpha_{mq}(t^{-1}h)\overline{\alpha_{lp}}(h)d\mu(\dot h)\ \text{according to [10]}\end{aligned}$$

$$= \frac{1}{d_\sigma} \sum_{r,s=1}^{d_\sigma} \int_{(G/K)^3} \alpha_{js}(t^{-1}g)\overline{\alpha}_{ir}(g)\alpha_{ms}(t^{-1}h)\overline{\alpha}_{lr}(h)d\mu(\dot{g})d\mu(\dot{h})d\mu(\dot{t})$$
$$= \frac{c_{ijlm}}{d_\sigma}.$$

If $\sigma \neq \tau$ then

$$\int_G u_{ij}^{L^\sigma}(t)\overline{u}_{lm}^{L^\tau}(t)d\lambda(t) = \int_{G/K} d\mu(\dot{t}) \sum_{r,s,p,q=1} \int_K (L_{rs}^\sigma(k^{-1})L_{pq}^\tau(k^{-1})d\nu(k)$$
$$\int_{G/K} \alpha_{js}(t^{-1}g)\overline{\alpha_{ir}}(g)d\mu(\dot{g}) \times$$
$$\times \int_{G/K} \alpha_{mq}(t^{-1}h)\overline{\alpha_{lp}}(h)d\mu(\dot{h}) = 0 \text{ according to [11]}$$

□

In the case of a particular orthonormal basis, the orthogonality relation reduces to the following:

Corollary 3.5 *Choosing an orthonormal basis* $(\xi_i)_{i=1}^{d\sigma}$ *of* H_σ *such that*

$$\int_{(G/K)^3} \alpha_{js}(t^{-1}g)\overline{\alpha}_{ir}(g)\alpha_{ms}(t^{-1}h)\overline{\alpha}_{lr}(h)d\mu(\dot{g})d\mu(\dot{h})d\mu(\dot{t}) = \begin{cases} \frac{1}{d_\sigma^2} & \text{if } j = m \text{ and } i = l \\ 0 & \text{if not} \end{cases} \tag{21}$$

(or, also, $c_{ijlm} = \delta_{il}\delta_{jm}$*) leads to*

$$\int_G u_{ij}^{L^\sigma}(t)\overline{u}_{lm}^{L^\sigma}(t)d\lambda(t) = \frac{\delta_{il}\delta_{jm}}{d_\sigma}. \tag{22}$$

Proof With the conditions 21, we have

$$\int_G u_{ij}^{L^\sigma}(t)\overline{u}_{lm}^{L^\sigma}(t)d\lambda(t) = \frac{1}{d_\sigma} \sum_{r,s=1}^{d_\sigma} \int_{(G/K)^3} \alpha_{js}(t^{-1}g)\overline{\alpha}_{ir}(g)\alpha_{ms}(t^{-1}h)\overline{\alpha}_{lr}(h)d\mu(\dot{g})d\mu(\dot{h})d\mu(\dot{t})$$
$$= \frac{\delta_{il}\delta_{jm}}{d_\sigma} \sum_{i,j=1}^{d_\sigma} \frac{1}{d_\sigma^2}$$
$$= \frac{\delta_{il}\delta_{jm}}{d_\sigma} \frac{d_\sigma^2}{d_\sigma^2}$$
$$\int_G u_{ij}^{L^\sigma}(t)\overline{u}_{lm}^{L^\sigma}(t)d\lambda(t) = \frac{\delta_{il}\delta_{jm}}{d_\sigma}.$$

In the sequel, $(\theta_i)_{i=1}^{\infty}$ will designate an orthonormal basis of H^{L^σ} and $(\xi_i)_{i=1}^{d\sigma}$ that of H_σ, where $(\xi_i)_{i=1}^{d\sigma}$ is chosen such that

$$\int_{G/K} d\mu(\dot{t}) \int_{G/K} \alpha_{js}(t^{-1}g)\overline{\alpha}_{ir}(g)d\mu(\dot{g}) \int_{G/K} \alpha_{ms}(t^{-1}h)\overline{\alpha}_{lr}(h)d\mu(\dot{h})$$
$$= \begin{cases} \frac{1}{d_\sigma^2} & \text{if } j = m \text{ and } i = l \\ 0 & \text{if not.} \end{cases}$$

□

3.2 Fourier–Stieltjes transform on a locally compact group with a compact subgroup

Definition 3.6 Assume $m \in M_1(G, \mathcal{A})$. We define the Fourier–Stieltjes transform of an arbitrary measure m as the family $(\hat{m}(\sigma))_{\sigma\in\Sigma}$ of sesquilinear mappings on $H^{L^\sigma} \times H^{L^\sigma}$ into $\mathcal{A}$, given by the relation

$$\hat{m}(\sigma)(u, v) = \int_G \left\langle \overline{U}_t^{L^\sigma} u, v \right\rangle_{H^{L^\sigma}} dm(t) = \int_G \int_{G/K} \left\langle \overline{u}(t^{-1}g), v(g) \right\rangle d\mu(\dot{g})dm(t). \quad (23)$$

According to the Remark 2.17 and the Definition 3.6, for any $f \in L_1(G, \mathcal{A}, \lambda)$, where λ denotes a Haar measure on G, the Fourier transform of f is a family $(\hat{f}(\sigma))_{\sigma\in\Sigma}$ of sesquilinear mappings of $H^{L^\sigma} \times H^{L^\sigma}$ into $\mathcal{A}$, given by the relation

$$\hat{f}(\sigma)(u, v) = \int_G \left\langle \overline{U}_t^{L^\sigma} u, v \right\rangle_{H^{L^\sigma}} f(t)d\lambda(t) = \int_G \int_{G/K} \left\langle \overline{u}(t^{-1}g), v(g) \right\rangle f(t)d\mu(\dot{g})d\lambda(t). \quad (24)$$

3.3 Properties of the Fourier–Stieltjes transform

Let denote by $\mathcal{S}(\Sigma, \mathcal{A}) = \prod_{\sigma\in\Sigma} \mathcal{S}(H^{L^\sigma} \times H^{L^\sigma}, \mathcal{A})$, where $\mathcal{S}(H^{L^\sigma} \times H^{L^\sigma}, \mathcal{A})$ is the set of all the sesquilinear mappings of $H^{L^\sigma} \times H^{L^\sigma}$ into $\mathcal{A}$. Also, assume $\prod_{\sigma\in\Sigma} \mathcal{S}(H^{L^\sigma} \times H^{L^\sigma}, \mathcal{A})$ is a vector space for the addition and multiplication by a scalar of mappings.

Definition 3.7 For $\Phi \in \mathcal{S}(\Sigma, \mathcal{A}) = \prod_{\sigma\in\Sigma} \mathcal{S}(H^{L^\sigma} \times H^{L^\sigma}, \mathcal{A})$, note $\|\Phi\|_\infty$ the quantity is defined by

$$\|\Phi\|_\infty = \sup\{\|\Phi(\sigma)\| : \sigma \in \Sigma\} \quad (25)$$

with $\|\Phi(\sigma)\| = \sup\{\|\Phi(\sigma)(u, v)\|_{\mathcal{A}} : \|u\|_{H^{L^\sigma}} \leq 1, \|v\|_{H^{L^\sigma}} \leq 1\}$. Let define:

(i) $\mathcal{S}_\infty(\Sigma, \mathcal{A}) = \{\Phi \in \mathcal{S}(\Sigma, \mathcal{A}) : \|\Phi\|_\infty < \infty\}$.
(ii) $\mathcal{S}_{00}(\Sigma, \mathcal{A}) = \{\Phi \in \mathcal{S}(\Sigma, \mathcal{A}) : \{\sigma \in \Sigma : \Phi(\sigma) \neq 0\}$ is finite$\}$.
(iii) $\mathcal{S}_0(\Sigma, \mathcal{A}) = \{\Phi \in \mathcal{S}(\Sigma, \mathcal{A}) : \forall\varepsilon > 0 \ \{\sigma \in \Sigma : \|\Phi(\sigma)\| > \varepsilon\}$ is finite$\}$.

Proposition 3.8 *Let us consider Φ in $\mathcal{S}(\Sigma, \mathcal{A})$. There is a matrix $(a_{ij}^\sigma)_{1\leq i,j\leq\infty}$, $a_{ij}^\sigma \in \mathcal{A}$ such that*

$$\Phi(\sigma) = \sum_{i,j=1}^{\infty} d_\sigma a_{ij}^\sigma \hat{u}_{ij}^{L^\sigma}$$

with $\hat{u}_{ij}^{L^\sigma}$ the Fourier transform of $u_{ij}^{L^\sigma}$ given by

$$\hat{u}_{ij}^{L^\sigma}(\sigma)(u,v) = \int_G \left\langle \overline{U}_t^{L^\sigma} u, v \right\rangle_{H^{L^\sigma}} u_{ij}^{L^\sigma}(t) d\lambda(t).$$

Proof For all $u, v \in H^{L^\sigma}$ such that $u = \sum_{j=1}^{\infty} \beta_j \theta_j$ and $v = \sum_{i=1}^{\infty} \gamma_i \theta_i$,

$$\Phi(\sigma)(u,v) = \sum_{i,j=1}^{\infty} \beta_j \overline{\gamma}_i \Phi(\sigma)(\theta_j, \theta_i).$$

Defining $(\Phi(\sigma)(\theta_j, \theta_i)) = (a_{ij}^\sigma)$, the matrix of $\Phi(\sigma)$ in the basis $(\theta_i)_{i=1}^{\infty}$ and we have

$$\begin{aligned}\hat{u}_{ij}^{L^\sigma}(\sigma)(u,v) &= \int_G \left\langle \overline{U}_t^{L^\sigma} u, v \right\rangle_{H^{L^\sigma}} u_{ij}^{L^\sigma}(t) d\lambda(t) \\ &= \sum_{l,m=1}^{\infty} \beta_m \overline{\gamma}_l \int_G \overline{u}_{lm}^{L^\sigma} u_{ij}^{L^\sigma} d\lambda(t) = \frac{\beta_j \overline{\gamma}_i}{d_\sigma}. \\ \Phi(\sigma)(u,v) &= \sum_{i,j=1}^{\infty} a_{ij}^\sigma \beta_j \overline{\gamma}_i = \sum_{i,j=1}^{\infty} d_\sigma a_{ij}^\sigma \hat{u}_{ij}^{L^\sigma}(u,v)\end{aligned}$$

which yields

$$\Phi(\sigma) = \sum_{i,j=1}^{\infty} d_\sigma a_{ij}^\sigma \hat{u}_{ij}^{L^\sigma}.$$

□

Corollary 3.9 *For any $f \in L_1(G, \mathcal{A}, \lambda)$, there is a matrix $(a_{ij})_{1 \le i,j \le \infty}$, $a_{ij} \in \mathcal{A}$ such that*

$$\hat{f}(\sigma) = \sum_{i,j=1}^{\infty} d_\sigma a_{ij}^\sigma \hat{u}_{ij}^{L^\sigma}.$$

Lemma 3.10 *Let us denote by $\mathcal{L}_\sigma(G)$ the set of finite linear combinations functions $t \longmapsto \left\langle U_t^{L^\sigma} u, v \right\rangle_{H^{L^\sigma}}$ and $\mathcal{L}(G) = \bigcup_{\sigma \in \Sigma} \mathcal{L}_\sigma(G)$. $\mathcal{L}(G)$ is dense in $\mathfrak{C}_0(G)$ for the topology of the uniform convergence, where $\mathfrak{C}_0(G)$ denotes the space of continuous funtions of G into $\mathbb{K}$ vanishing at infinity.*

Proof (i) First, we have $\mathcal{L}(G) \subset \mathfrak{C}_0(G)$ moreover for $f \in \mathcal{L}(G)$ we have abviously $\overline{f} \in \mathcal{L}(G)$.

(ii) Let t_1 and t_2 be two elements of G such that $t_1 \neq t_2$. Let us suppose by contradiction that for every $f \in \mathcal{L}(G)$, $f(t_1) = f(t_2)$. Thus, by chosing $f = f_k = u_{ik}^{L^\sigma}$ we have $\left\langle U_{t_1}^{L^\sigma} \theta_i, \theta_k \right\rangle = \left\langle U_{t_2}^{L^\sigma} \theta_i, \theta_k \right\rangle$ for $i, k \in \{1, 2, ...\}$. Let us fix i and make k running over the set $\{1, 2, ...\}$. Then, antilinear mappings $\phi_{U_{t_1}^{L^\sigma} \theta_i} = \left\langle U_{t_1}^{L^\sigma} \theta_i, . \right\rangle$ and $\phi_{U_{t_2}^{L^\sigma} \theta_i} = \left\langle U_{t_2}^{L^\sigma} \theta_i, . \right\rangle$ are identically equal in H^{L^σ} therefore $U_{t_1}^{L^\sigma} \theta_i = U_{t_2}^{L^\sigma} \theta_i \Longrightarrow \theta_i = U_{t_2 t_1^{-1}}^{L^\sigma} \theta_i \Longrightarrow U_{t_2 t_1^{-1}}^{L^\sigma} \equiv I_{H^{L^\sigma}}$. It means that $t_2 = t_1$. There is a contradiction; hence, there is $f \in \mathcal{L}(G)$ such that $f(t_1) \neq f(t_2)$.

(iii) Let t be any element of G; since $U_t^{L^\sigma}$ is invertible, there exist $i, k \in \{1, 2, ...\}$ such that $U_t^{L^\sigma} \theta_i = \theta_k$. Therefore, we have $\left\langle U_t^{L^\sigma} \theta_i, \theta_k \right\rangle = \langle \theta_k, \theta_k \rangle = 1 \neq 0$ i.e $f(t) \neq 0$.

According to Stone-Weierstrass theorem for locally compact spaces $\mathcal{L}(G)$ is dense in $\mathfrak{C}_0(G)$. □

Theorem 3.11 *The mapping* $m \longmapsto \hat{m}$ *from* $M_1(G, \mathcal{A})$ *into* $\mathcal{S}_\infty(\Sigma, \mathcal{A})$ *is linear, injective and continuous.*

Proof Let $m, n \in M_1(G, \mathcal{A})$ such that $\hat{m} = \hat{n}$. For any $u, v \in H^{L^\sigma} \times H^{L^\sigma}$ and any $\sigma \in \Sigma$, we have:

$$\begin{aligned} \hat{m} = \hat{n} &\iff \int_G \left\langle \overline{U_t^{L^\sigma}} u, v \right\rangle_{H^{L^\sigma}} dn(t) = \int_G \left\langle \overline{U_t^{L^\sigma}} u, v \right\rangle_{H^{L^\sigma}} dm(t) \\ &\iff \int_G \left\langle \overline{U_t^{L^\sigma}} u, v \right\rangle_{H^{L^\sigma}} d(n-m)(t) = 0 \end{aligned} \tag{26}$$

for any σ in Σ, and u,v in H^{L^σ}.

$\mathcal{L}(G)$ is dense in $\mathfrak{C}_0(G)$ according to the previous lemma. Then, $\mathcal{L}(G)$ is dense in $\mathcal{K}(G)$. Thus $n - m$ can be viewed as a linear map of $\mathcal{K}(G)$ which is identically null. Then $\int_G \left\langle \overline{U_t^{L^\sigma}} u, v \right\rangle_{H^{L^\sigma}} d(n-m)(t) = 0 \Longrightarrow n - m \equiv 0$ i.e. $m = n$. The mapping $m \longmapsto \hat{m}$ is therefore injective.

Next, let us now prove the continuity of the mapping $m \longmapsto \hat{m}$.

$$\begin{aligned} \|\hat{m}(\sigma)\| &= \sup\left\{ \|\hat{m}(\sigma)(u,v)\|_{\mathcal{A}} : \|u\|_{H^{L^\sigma}}, \leq 1 \text{ and } \|v\|_{H^{L^\sigma}}, \leq 1 \right\} \\ &= \sup\left\{ \left\| \int_G \left\langle \overline{U_t^{L^\sigma}} u, v \right\rangle_{H^{L^\sigma}} dm(t) \right\|_{\mathcal{A}} : \|u\|_{H^{L^\sigma}}, \leq 1 \text{ and } \|v\|_{H^{L^\sigma}}, \leq 1 \right\} \\ &\leq \int_G \chi_G d|m| = \|m\|, \end{aligned}$$

since $U_t^{L^\sigma}$ is unitary. Thus, $\|\hat{m}(\sigma)\| \leq \|m\|$, $\sigma \in \Sigma$ and $\|\hat{m}\|_\infty \leq \|m\|$. As a consequence, $\hat{m} \in \mathcal{S}_\infty(\Sigma, \mathcal{A})$ and the mapping is continuous.

Corollary 3.12 *The mapping* $f \longmapsto \hat{f}$ *from* $L_1(G, \lambda, \mathcal{A})$ *into* $\mathfrak{S}_\infty(\Sigma, \mathcal{A})$ *is linear, injective, and continuous.*

Proof Here we consider $f, g \in L_1(G, \lambda, \mathcal{A})$ such that $\hat{f} = \hat{g}$ then

$$\begin{aligned} \int_G \left\langle \overline{U_t^{L^\sigma}} u, v \right\rangle_{H^{L^\sigma}} f d\lambda(t) &= \int_G \left\langle \overline{U_t^{L^\sigma}} u, v \right\rangle_{H^{L^\sigma}} f d\lambda(t) \\ &\Longrightarrow \int_G \left\langle \overline{U_t^{L^\sigma}} u, v \right\rangle_{H^{L^\sigma}} (f-g) d\lambda(t) = 0. \end{aligned}$$

For the same reasons as the previous proof we have $f - g \equiv 0$ then $f = g$.

$$\begin{aligned} \|\hat{f}(\sigma)\| &= \sup\left\{ \|\hat{f}(\sigma)(u,v)\|_{\mathcal{A}} : \|u\|_{H^{L^\sigma}}, \leq 1 \text{ and } \|v\|_{H^{L^\sigma}}, \leq 1 \right\} \\ &= \sup\left\{ \left\| \int_G \left\langle \overline{U_t^{L^\sigma}} u, v \right\rangle_{H^{L^\sigma}} f(t) d\lambda(t) \right\| : \|u\|_{H^{L^\sigma}}, \leq 1 \text{ and } \|v\|_{H^{L^\sigma}}, \leq 1 \right\} \\ &\leq \int_G \|f(t)\| d\lambda(t) = \|f\|_1. \end{aligned}$$

Thus, $\|\hat{f}(\sigma)\| \leq \|f\|_1, \sigma \in \Sigma$ and $\|\hat{f}\|_\infty \leq \|f\|_1$. Hence, $\hat{f} \in \mathfrak{S}_\infty(\Sigma, \mathcal{A})$ and the mapping is continuous. □

3.4 Modified Peter–Weyl theorem

The following theorem gives the inverse formula of Fourier transform in the context of our work.

Theorem 3.13 *For every $f \in L_2(G, \mathcal{A})$, there is $a_{ij}^{\sigma} \in \mathcal{A}$, $1 \le i, j < \infty$, $\sigma \in \Sigma$ such that*

$$f = \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{\infty} a_{ij}^{\sigma} u_{ij}^{L^\sigma}. \tag{27}$$

Proof With Proposition 3.8 for $g \in L_1(G, \lambda, \mathcal{A})$, there exists $(a_{ij}^{\sigma})_{1\le i,j\le\infty}$, $a_{ij}^{\sigma} \in \mathcal{A}$ such that

$$\hat{g}(\sigma) = \sum_{i,j=1}^{\infty} d_\sigma a_{ij}^{\sigma} \hat{u}_{ij}^{L^\sigma}$$

with $\hat{u}_{ij}^{L^\sigma}(\sigma)(u, v) = \int_G \left\langle \overline{U}_t^{L^\sigma} u, v \right\rangle_{H^{L^\sigma}} u_{ij}^{L^\sigma}(t) d\lambda(t)$ and $\hat{g}(\sigma)(\theta_j, \theta_i) = a_{ij}^{\sigma}$.
Thus $\hat{g} = \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{\infty} a_{ij}^{\sigma} \hat{u}_{ij}^{L^\sigma}$.

It is well known that $L_1(G, \lambda, \mathcal{A}) \cap L_2(G, \lambda, \mathcal{A})$ is dense in $L_2(G, \lambda, \mathcal{A})$. Then, for every $f \in L_2(G, \lambda, \mathcal{A})$ there exists a sequence (f_n) in $L_1(G, \lambda, \mathcal{A}) \cap L_2(G, \lambda, \mathcal{A})$ such that $f_n \xrightarrow[n\to\infty]{} f$ in $L_2(G, \lambda, \mathcal{A})$. $\hat{f}_n = \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1^\infty}^{n} a_{ij}^{\sigma} \hat{u}_{ij}^{L^\sigma}$ where ${}^n a_{ij}^{\sigma} = \hat{f}_n(\sigma)(\theta_j, \theta_i)$. We have $\hat{f}_n(\sigma)(\theta_j, \theta_i) \xrightarrow[n\to\infty]{} \hat{f}(\sigma)(\theta_j, \theta_i)$ ie there exists a_{ij}^{σ} in $\mathcal{A}$ such that ${}^n a_{ij}^{\sigma} \xrightarrow[n\to\infty]{} a_{ij}^{\sigma}$. Finally $\hat{f} = \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{\infty} a_{ij}^{\sigma} \hat{u}_{ij}^{L^\sigma}$.

According to the Corollary 3.12 the mapping $f \longmapsto \hat{f}$ is injective; as a consequence,

$$\hat{f} = \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{\infty} a_{ij}^{\sigma} \hat{u}_{ij}^{L^\sigma} \text{ i.e. } \hat{f} = \widehat{\left(\sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{\infty} a_{ij}^{\sigma} u_{ij}^{L^\sigma} \right)} \Longrightarrow f = \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{\infty} a_{ij}^{\sigma} u_{ij}^{L^\sigma}. \quad \square$$

Corollary 3.14 *For every $f \in L_2(G, \mathcal{A})$, we have*

$$f = \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{\infty} \hat{f}(\theta_j, \theta_i) u_{ij}^{L^\sigma}. \tag{28}$$

4 Concluding remarks

In this study, we've formulated a comprehensive framework for the Fourier–Stieltjes transform, complete with Shur's orthogonality property, tailored specifically for locally compact groups. Our contributions encompass:

- Establishing the Fourier–Stieltjes transform of a bounded vector measure on a locally compact group within the confines of a Banach algebra.
- Unraveling the Fourier transform of a Haar integrable function.
- Delving into the pertinent properties associated with our newly developed Fourier–Stieltjes transform. These properties encompass sesquilinearity, injectivity, and continuity, each playing a crucial role in understanding the transform's behavior and applications.

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