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Computational and Applied Mathematics

Revisiting von Mises Plasticity with Convex Thermodynamic Potentials for Enhanced Numerical Precision and Stability

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Contents

Abstract

This paper investigates the relationship between the von Mises plasticity model with isotropic hardening and the class of generalized standard materials, characterized by a reversible elastic energy density and a dissipation potential. Our analysis demonstrates that within the linearized theory framework, the constitutive equations of the von Mises plasticity model indeed define a generalized standard material. We further discuss the implications of this finding for the robustness of the projection algorithm used in the finite element implementation of the model. The results contribute to a deeper understanding of the model's mathematical foundations and its computational stability in numerical simulations.

Keywords: Generalized standard materials, von Mises model, Plasticity, Projection problem

1. Introduction

The formalism of generalized standard materials (GSM) was first introduced by Halphen and Nguyen (1975) in the context of elasto-plasticity. In short, the constitutive equations of GSM are described by the expressions of the elastic energy density (also called reversible elastic potential) and the dissipation potential. The elastic energy density, stored in the material by its deformation under external stimuli, provides by derivation the Cauchy stress and the thermodynamics forces, while the dissipation potential, gives rise to the evolution equations of the internal states variables. The formalism GSM was developed within the framework of linearized theory and is suitable for rate independent materials. Although these two restrictions may limit the use of this formalism to describe a vast majority of materials and their behaviors under various external conditions, the successes credited to the GSM approach is tremendous in view of the nice local and global stability properties it offers for robust numerical implementations in finite element subroutines. Of course, for other materials, additional precautions are required to guarantee local and global stabilities of numerical schemes, since the tangent stiffness matrix can become non-invertible during nonlinear analyses.

Initially developed in the context of small strain rate independent elasto-plasticity by Halphen and Nguyen (1975) and later on reviewed by Ziegler and Wehrli (1987), the GSM formalism was extended to finite strain elasto-plasticity by Hackl (1997). The later extension was proposed as an alternative theoretical framework to overcome the usual problems encountered in the use of classical finite elasto-plasticity models. These problems include arbitrariness in the choice of yield functions and flow rules, difficulty to obtain a clear distinction between the concept of frame indifference and material symmetry which is complicated by the unclear role played by the introduction of the intermediate configuration[1], and generally non-equivalence between the yield functions obtained in the different configurations introduced by the adoption of a multiplicative decomposition of the deformation gradient, see Lee (1969). Once developed, the GSM framework has continuously played a key role in modeling of materials, see Fremond (2002),

[1]Some authors may argued that the velocity gradient can be additively decomposed into an elastic and plastic parts *only* in the intermediate configuration; however, from the author point of view, this can simply be a heuristic assumption from the beginning, exactly as in the small strain formulation where the Eularian deformation rate is additively decomposed into an elastic and a plastic part

Hackl (1997), Maugin (1992) among other authors, and ductile fracture in porous solids, see Enakoutsa et al. (2007). Applications of the formalism to metal plasticity whether it is described by a simple von Mises model within the linearized theory assumption or a J_2 plasticity theory has received little attention in the litterature. The objective of this work is to overcome this insufficiency. Namely, we shall study the relationship between the formalism GSM and a very simple metal plasticity model, the von Mises model. The outline of the paper is as follows.

- Section 2 gives an overview of some of the work of Halphen and Nguyen (1975) and Nguyen (1977) on generalized standard materials.
- Section 3 demonstrates how the constitutive relations of the classical small strain von Mises model with isotropic hardening define a generalized standard material. In addition, we discuss the benefits of this property on the numerical implementation of von Mises model into a numerical subroutine using the well-known projection algorithm.

2. Overview of the formalism of GSM

In this section we give a brief summary of some aspects of the works of Halphen and Nguyen (1975) and Nguyen (1977) on GSM. The theory is applicable only in the context of linearized theory.

The constitutive law of a generalized standard material is described by two thermodynamic potentials. The first one is the free energy $\psi(\boldsymbol{\epsilon}, \boldsymbol{\alpha})$, which is a function of the strain tensor $\boldsymbol{\epsilon}$ and a familly of internal variables collectively denoted $\boldsymbol{\alpha}$. This function must be convex with respect to both the varibles $\boldsymbol{\epsilon}$ and $\boldsymbol{\alpha}$ taken separately. (Convexity with respect to the global variable $(\boldsymbol{\epsilon}, \boldsymbol{\alpha})$ is not required). The stress tensor $\boldsymbol{\sigma}$ and the thermodynamic force $\mathbf{F}$ associated to $\boldsymbol{\alpha}$ are then given by

$$\boldsymbol{\sigma} \equiv \frac{\partial \psi}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \boldsymbol{\alpha}) \quad ; \quad \mathbf{F} \equiv -\frac{\partial \psi}{\partial \boldsymbol{\alpha}}(\boldsymbol{\epsilon}, \boldsymbol{\alpha}). \tag{1}$$

The second thermodynamic potential is the dissipation potential $\phi(\dot{\boldsymbol{\alpha}})$. This function must be convex, non-negative and zero for $\dot{\boldsymbol{\alpha}} = \mathbf{0}$. It governs the evolution equations of the internal variables through the equivalent equations

$$\mathbf{F} \in \partial \phi(\dot{\boldsymbol{\alpha}}) \quad \Leftrightarrow \quad \dot{\boldsymbol{\alpha}} \in \partial \tilde{\phi}(\mathbf{F}) \tag{2}$$

where $\tilde{\phi}$ denotes the Legendre-Fenchel transform of ϕ, and $\partial\phi$ and $\partial\tilde{\phi}$ the sub-differentials of ϕ and $\tilde{\phi}$. In the case of a time-independent behavior, as considered in this paper, ϕ is a positively homogeneous function of degree 1 of $\dot{\boldsymbol{\alpha}}$. Its Legendre-Fenchel transform $\tilde{\phi}$ is then the indicator function of a closed convex set $\mathcal{C}$ (the *reversibility domain*) in the space of thermodynamic forces $\mathbf{F}$. This set being defined by an inequality of the type $\Phi(\mathbf{F}) \leq 0$ for some function Φ, the sub-differential $\partial\tilde{\phi}(\mathbf{F})$ consists of the sole vector $\mathbf{0}$ if $\mathbf{F}$ lies in the interior of $\mathcal{C}$, of the half-straight line $\{\eta(\partial\Phi/\partial\mathbf{F})(\mathbf{F}),\ \eta \geq 0\}$ if $\mathbf{F}$ lies on the boundary of $\mathcal{C}$, and is empty if $\mathbf{F}$ lies outside $\mathcal{C}$. The evolution law $(2)_2$ of $\boldsymbol{\alpha}$ may thus be re-written in the equivalent form

$$\dot{\boldsymbol{\alpha}} = \eta \frac{\partial\Phi}{\partial\mathbf{F}}(\mathbf{F}) \quad , \quad \eta \left\{ \begin{array}{lll} = 0 & \text{if} & \Phi(\mathbf{F}) < 0 \\ \geq 0 & \text{if} & \Phi(\mathbf{F}) = 0. \end{array} \right. \tag{3}$$

This means that the evolution of $\boldsymbol{\alpha}$ obeys a kind of "generalized normality property".

Generalized standard materials obey several nice properties. The first one is that the evolution equation $(2)_2$ of $\boldsymbol{\alpha}$ automatically warrants *non-negativeness of the dissipation* $\mathcal{D}$, and thus thermodynamic consistency of the model.

The second property is given as follows. Let quantities at time t be denoted with an upper index ^{0} and quantities at $t + \Delta t$ without any special symbol. Then, provided that the evolution equation (2) of $\boldsymbol{\alpha}$ is discretized in time with an implicit scheme, the determination of the value of $\boldsymbol{\alpha} \equiv \boldsymbol{\alpha}(t+\Delta t)$ from those of $\boldsymbol{\epsilon}^0 \equiv \boldsymbol{\epsilon}(t)$, $\boldsymbol{\alpha}^0 \equiv \boldsymbol{\epsilon}(t)$ and $\Delta\boldsymbol{\epsilon} \equiv \boldsymbol{\epsilon}(t + \Delta t) - \boldsymbol{\epsilon}(t) \equiv \boldsymbol{\epsilon} - \boldsymbol{\epsilon}^0$ (projection problem) is equivalent to minimizing the function

$$\chi(\boldsymbol{\epsilon}, \boldsymbol{\alpha}^0, \Delta\boldsymbol{\alpha}) \equiv \psi(\boldsymbol{\epsilon}, \boldsymbol{\alpha}^0 + \Delta\boldsymbol{\alpha}) + \phi(\Delta\boldsymbol{\alpha}) \tag{4}$$

with respect to $\Delta\boldsymbol{\alpha}$.

The third property which is a consequence of the second one, is that of *symmetry of the tangent matrix of the global elasto-plastic iterations.*

The proofs of the three previous properties were widely discussed in Enakoutsa et al. (2007) in the context of ductile fracture of porous solids and are not repeated here.

3. von Mises model and the class of GSM

The objective of this section is to demonstrate that the constitutive relations of von Mises plasticty model with isotropic hardening define a generalized standard material. We begin by recalling the constitutive equations of this model which consist of several elements.

- The first element, the von Mises yield criterion with isotropic hardening, reads

$$\sigma_{eq} \equiv \left[\frac{3}{2}\sigma'_{ij}\sigma'_{ij}\right]^{\frac{1}{2}} \leq \sigma(\varepsilon_{eq}) = \bar{\sigma}. \tag{5}$$

  In this equation σ_{eq} denotes the "von Mises equivalent stress", $\boldsymbol{\sigma}'$ is the deviatoric stress tensor, and $\sigma(\varepsilon_{eq})$ represents the yield stress in simple tension which depends on the "von Mises cumulative equivalent plastic strain" ε_{eq} defined by:

$$\varepsilon_{eq} \equiv \int_0^t \dot{\varepsilon}_{eq}(r)\mathrm{d}r \quad , \quad \dot{\varepsilon}_{eq} \equiv \left[\frac{2}{3}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}\right]^{\frac{1}{2}} \tag{6}$$

  where $\dot{\varepsilon}_{eq}$ is the "von Mises equivalent plastic strain rate". The parameter $\sigma(\varepsilon_{eq})$ is usually determined experimentally by means of simple tension tests. For the sake of simplicity, we idealize this function by a linear formula in the form:

$$\sigma(\varepsilon_{eq}) = \sigma_0 + h\varepsilon_{eq} \tag{7}$$

  where σ_0 represents the initial (obtained before the appearance of any strain hardening) yield stress in simple tension tests, and h is a positive hardening slope.

- The second element is the Prandlt-Reuss flow rule which obeys the "normality rule" and is defined as:

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{3}{2}\frac{\dot{\varepsilon}_{eq}}{\sigma_{eq}}\boldsymbol{\sigma}', \quad \dot{\varepsilon}_{eq} \left\{ \begin{array}{ccll} = & 0 & \text{if} & \sigma_{eq} < \sigma(\varepsilon_{eq}) \\ \geq & 0 & \text{if} & \sigma_{eq} = \sigma(\varepsilon_{eq}). \end{array} \right. \tag{8}$$

We shall now show that these equations satisfy the required properties to fit in the class of generalized standard materials. To that end, we must first define the state variables and the free energy or the elastic potential of the material, then check that the later meets the required properties defined in Section 2.

The state of the material is described by the following variables: the components of the total strain $\boldsymbol{\varepsilon}$ and a set of internal variables including the components of the plastic deformation $\boldsymbol{\varepsilon}^p$ and the mean equivalent plastic strain. The free energy is defined as an elastic deformation energy plus a "blocked" strain hardening energy

$$\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \varepsilon_{eq}) = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \int_0^{\varepsilon_{eq}} \sigma(\varepsilon)\mathrm{d}\varepsilon \tag{9}$$

where $\mathbf{C}$ is the fourth-rank elastic stiffness matrix and $\sigma(\varepsilon_{eq})$ is the yield stress which depends upon the cumulative plastic strain.

With this definition, it is obvious that the free energy ψ is *strictly convex* with respect to the internal variable $\boldsymbol{\varepsilon}$, the quadratic form defined by $\mathbf{C}$ being positive-definite. The free energy is also strictly convex with respect to $\boldsymbol{\varepsilon}^p$ for the same reason, as previously invoqued. Thanks to the fact that the hardening slope is positive, the free energy is also strictly convex with respect to the variable ε_{eq}. Furthermore, the free energy is the sum of two strictly convex functions depending upon $\boldsymbol{\varepsilon}^p$ and ε_{eq}; consequently, the free energy is strictly convex with respect to the global variable $(\boldsymbol{\varepsilon}^p, \varepsilon_{eq})$ as desired.

The derivative of ψ with respect to $\boldsymbol{\varepsilon}$ is equal to $\boldsymbol{\sigma}$, as also desired, and the thermodynamic forces $\mathbf{F}^{\boldsymbol{\varepsilon}^p}$ and $F^{\varepsilon_{eq}}$, associated to the internal variables $\boldsymbol{\varepsilon}^p$ and ε_{eq}, are given by

$$\begin{cases} \mathbf{F}^{\boldsymbol{\varepsilon}^p} = -\dfrac{\partial\psi}{\partial\boldsymbol{\varepsilon}^p} = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) = \boldsymbol{\sigma} \\ \\ F^{\varepsilon_{eq}} = -\dfrac{\partial\psi}{\partial\varepsilon_{eq}} = -\sigma(\varepsilon_{eq}) \equiv \sigma \end{cases} \tag{10}$$

(by definition of the current yield stress $\bar{\sigma}$.)

The next task to complete is to check that the reversibility domain defined by von Mises yield criterion with isotropic hardening in the space of thermodynamic forces (by expressing von Mises yield function as a function of the variables $\mathbf{F}^{\boldsymbol{\varepsilon}^p}$ and $F^{\varepsilon_{eq}}$, instead of the variables $\boldsymbol{\sigma}$ and σ) is convex. The transformation of the variables $(\boldsymbol{\sigma},\, \sigma)$ to $(\mathbf{F}^{\boldsymbol{\varepsilon}^p},\, F^{\varepsilon_{eq}}) = (\boldsymbol{\sigma}, -\sigma)$ being linear, it suffices to show that the reversibility domain in the space of the first variables, $\mathcal{C} \equiv \{(\boldsymbol{\sigma}, \sigma); \quad \Phi(\boldsymbol{\sigma}, \sigma) \leq 0\}$, is convex. This is obvious due to the fact that von Mises yield function Φ [2] is a convex function with respect to the global variable $(\boldsymbol{\sigma},\, \sigma)$. Indeed,

$$\Phi(\boldsymbol{\sigma}, \bar{\sigma}) = \Phi(\mathbf{F}^{\boldsymbol{\varepsilon}^p}, \mathrm{F}^{\varepsilon_{eq}}) = \sigma_{eq} - \bar{\sigma} = ||\mathrm{F}^{\boldsymbol{\varepsilon}^{p\prime}}|| + \mathrm{F}^{\varepsilon_{eq}} \tag{11}$$

where the symbol $||.||$ denotes the Eucludian norm. It follows that von Mises yield function Φ is convex; hence, by linearity of the transformation of the

[2] The expression of the yield criterion (5) allows to define such a function

variables ($\boldsymbol{\sigma}$, $\bar{\sigma}$) to ($\mathrm{F}^{\boldsymbol{\varepsilon}^p}$, $\mathrm{F}^{\varepsilon_{eq}}$)=($\boldsymbol{\sigma}, -\bar{\sigma}$), the reversibility domain is convex with respect to the global variable ($\boldsymbol{\sigma}$, $\bar{\sigma}$).

The last thing to check is that the evolution equations associated to the internal variables $\boldsymbol{\varepsilon}^p$ and ε_{eq} satisfy the "generalized normality rule" with respect to von Mises yield function, expressed as a function of the thermodynamic forces, i.e.:

$$\left\{ \begin{array}{lcl} \dot{\boldsymbol{\varepsilon}}^p & = & \eta \dfrac{\partial \Phi}{\partial \mathbf{F}^{\boldsymbol{\varepsilon}^p}} \equiv \eta \dfrac{\partial \Phi}{\partial \boldsymbol{\sigma}} \\ \\ \dot{\varepsilon}_{eq} & = & \eta \dfrac{\partial \Phi}{\partial F^{\varepsilon_{eq}}} \equiv -\eta \dfrac{\partial \Phi}{\partial \sigma} \end{array} \right. \tag{12}$$

Note that the evolution equation $(12)_1$ is equivalent to the flow rule associated to the yield criterion by the normality property. It suffices, to complete the verification, to check that the evolution equation $(12)_2$ is satisfied. And yet

$$\Phi(\boldsymbol{\sigma}, \sigma) = \sigma_{eq} - \sigma \quad \Rightarrow \quad \frac{\partial \Phi}{\partial \sigma_{ij}} = \frac{3}{2}\frac{{\sigma_{ij}}'}{\sigma_{eq}} \quad \text{and} \quad \frac{\partial \Phi}{\partial \sigma} = -1.$$

The relation $(12)_1$ then gives

$$\dot{\varepsilon}^p_{ij} = \eta \frac{3}{2}\frac{\sigma'_{ij}}{\sigma_{eq}}. \tag{13}$$

Taking the magnitude of both sides of Eq.(13), we get

$$\eta = \dot{\varepsilon}_{eq},$$

which is precisely the value of η given by Eq.$(12)_2$. Hence, the "generalized normality rule" with respect to the global variable $(\boldsymbol{\varepsilon}, \varepsilon_{eq})$ is satisfied. This proves that small strain von Mises plasticity model with isotropic hardening can be described within the context of generalized standard materials, which guarantees that this model is automatically thermodynically consistent. From the numerical point of view, the generalized standard character of von Mises model ensures that the tangent matrix associated to the global elasto-plastic iterations is symmetric; this should avoid spurious problems of non-invertible matrix arrising during nonlinear analyses. The property also warrants that the problem of projection of the elastically computed stress tensor onto the yield locus (plastic correction of the elastic predictor) admits a unique solution, provided that the equations of this problem are obtained through implicit time-discretization with respect to the components of the plastic strain and the hardening parameter.

4. Closure

In conclusion, we have proposed a thermodynamic re-formulation of the constitutive equations governing the von Mises plasticity model within the framework of small strain theory. This re-formulation utilizes two key potentials: a convex elastic energy density to describe the material's elastic state and an irreversible plastic dissipation potential to govern the evolution of plasticity.

The stress and state variables are derived from the elastic energy, while the plastic dissipation potential provides the evolution equations for the internal state variables. Our analysis reveals that the reversibility domain associated with this formulation is convex, and the thermodynamic forces comply with a "generalized normality rule."

Consequently, the von Mises plasticity model with isotropic hardening can be effectively described within the context of generalized standard materials. This ensures robust local and global stability in numerical implementations, which is crucial for achieving high-fidelity structural designs involving elasto-plastic materials in finite element analysis.

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