Applied Differential Equations

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper.
Write your university identification number at the top of each sheet of paper.

DO NOT WRITE YOUR NAME!
Complete this sheet and staple to your answers. Read the directions of the exam very carefully.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE
****

1. ________________________ 5. ____________________________

2. ________________________ 6. ____________________________

3. ________________________ 7. ____________________________

4. ________________________ 8. ____________________________

Pass/fail recommend on this form.

Total score: _____________________

ADE Exam, Fall 2024
Department of Mathematics, UCLA

1. Consider the following initial value problem for $\theta(t) \in \mathbb{T}$ (The one dimensional torus, i.e. $[0, 2\pi)$ with periodic boundary conditions) given by

$$\begin{aligned} &\theta'' + \alpha\theta' + \frac{g}{\ell}\sin\theta(t) = 0 \\ &\theta(0) = \theta_0 \\ &\theta'(0) = \omega_0, \end{aligned}$$

where $\alpha, g, \ell > 0$ are positive constants such that $\alpha^2\ell < 4g$.

(a) Re-write this as a first order system in $(\theta, \theta') \in \mathbb{T} \times \mathbb{R}$.

(b) Find all of the equilibrium points, compute the linearizations, and classify the equilibria as linearly stable, linearly unstable, or linear centers.

(c) Prove that $H = \frac{1}{2}\theta'^2 - \frac{g}{\ell}\cos\theta$ is a Lyapunov function.

(d) Prove directly (i.e. don't just quote a theorem) that the origin is an asymptotically stable fixed point for the nonlinear problem (you don't have to use H).

2. (a) Derive power-series representations of the two linearly independent solutions of the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \nu^2\right)y = 0 \tag{1}$$

for $x > 0$, with ν a non-integer, non-zero real number (assume wlog that $\nu > 0$).

(b) When ν is a non-zero integer, explain how to derive the second solution and obtain a recurrence relation for the coefficients of this solution. (You do not need to solve for the coefficients.)

3. Consider the differential equation

$$u_t = -\mu\Delta u - \Delta^2 u$$

on $\mathbb{T}^2$, the periodic 2-torus $[0, 2\pi)^2$. (a) if $\mu = 2$ find a solution whose amplitude increases as t increases. (b) Find a value of μ_0 so that the solution is globally bounded in time for all $\mu < \mu_0$.

4. Let D be a bounded bomain in $\mathbb{R}^3$ with smooth boundary ∂D. Show that a solution of the boundary value problem

$$\Delta^2 u = f \quad \text{in} \quad D, \quad u = \Delta u = 0 \quad \text{on} \quad \partial D$$

must be unique.

5. Given a function $f \in H^2(\mathbb{T}^N)$ ($\mathbb{T}^N$ is the N-torus, periodic in N dimensions), and a smooth function $G : \mathbb{R} \to \mathbb{R}$, consider the energy

$$E(u) = \int_{\mathbb{T}^N} G(\Delta u) + \lambda \int_{\mathbb{T}^N} (f-u)^2.$$

(a) Derive the Euler-Langrange equation for extrema of E both in (i) weak form ($u \in H^2(\mathbb{T}^N)$) and in (ii) strong form (assuming u is smooth).

(b) Show that $E(u)$ is strictly convex if G is a strictly convex function on the line.

6. Let

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right), \; x \in \mathbb{R}\,, \; t > 0 \tag{2}$$

be the fundamental solution of the one-dimensional diffusion equation

$$u_t = \alpha \Delta u\,. \tag{3}$$

Consider

$$u(x,t) = \begin{cases} \int_{-\infty}^{\infty} \Phi(x-y) g(y)\, dy\,, & t > 0 \\ g(x)\,, & t = 0\,, \end{cases} \tag{4}$$

where $g(x)$ is a bounded, continuous, integrable function on $\mathbb{R}$.

Prove that $u(x,t)$ is a C^∞ solution of (3) on $\{(x,t) | t > 0\}$. Additionally, show that

$$\lim_{t \to 0} u(x,t) = g(x) \tag{5}$$

for all $x \in \mathbb{R}$.

7. Consider the equation

$$u_{x_1} u_{x_2} = u \;\text{ on }\; \Omega = \{(x_1, x_2) | x_1 > 0]\}\,, \;\text{ with }\; u(0, x_2) = x_2^2\,. \tag{6}$$

(a) Provide an argument for why the solution should be a polynomial that is homogeneous of degree two.

(b) Solve the equation with the given boundary condition using either the method of characteristics or by using the structure of the solution that you argued in part (a).

8. (a) Determine the solution $u(x,t)$ of

$$\begin{aligned} u_{tt} - 9u_{xx} &= \frac{18}{t^2+1}\,, \quad x \in \mathbb{R}\,,\ t \in \mathbb{R}\,, \\ u(x,0) &= x^2\,,\ u_t(x,0) = 9x\,. \end{aligned} \tag{7}$$

(b) Suppose that (7) is satisfied only for $x \in (0,6)$. In what region is $u(x,t)$ uniquely defined? Draw this region in the (x,t) plane.