ANALYSIS QUALIFYING EXAM: SEPTEMBER 19, 2024

In order to pass the exam, you need to show mastery of both real and complex analysis. Choose ten problems to work on, including five from Problems 1–6 and five from Problems 7–12. Please indicate on the front of your paper which ten problems you wish to have graded.

Problem 1. Let $f \in L^1(\mathbb{R})$ and suppose $g \in L^\infty(\mathbb{R})$ is periodic with period $T > 0$. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $a_n \to +\infty$ as $n \to \infty$. Show that

$$\int_{\mathbb{R}} f(x)\, g(a_n x + b_n)\, dx \to \left(\int_{\mathbb{R}} f(x)\, dx\right)\left(\frac{1}{T}\int_0^T g(y)\, dy\right) \quad \text{as } n \to \infty.$$

Hint: Assume first that $f \in C_0(\mathbb{R})$.

Problem 2. Let $f \in L^1(\mathbb{R})$ and let $\{a_n\}$ be a bounded sequence in $\mathbb{R}$. Assume that

$$g(x) = \lim_{n\to\infty} f(x - a_n)$$

exists for almost all $x \in \mathbb{R}$ and that it is not the case that $g(x) = 0$ almost everywhere. Show that the sequence $\{a_n\}$ converges.

Problem 3. Let $\tau \in \mathbb{R}$ and let $E_\tau \subseteq \mathbb{R}^2$ be the set of all $x = (x_1, x_2) \in \mathbb{R}^2$ for which there exists $C_x > 0$ such that

$$|x \cdot \lambda| = |x_1\lambda_1 + x_2\lambda_2| \geq \frac{C_x}{|\lambda|^\tau},$$

for all $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 \backslash \{0\}$. Show that E_τ is a set of full measure in $\mathbb{R}^2$ if $\tau > 1$.

Problem 4. Let $0 \leq f_n \in L^1(\mathbb{R}^d)$ for $n = 1, 2, \ldots$ be such that $\int_{\mathbb{R}^d} f_n\, dx = 1$ and $\int_K f_n\, dx \to 0$ as $n \to \infty$ for each compact set $K \subseteq \mathbb{R}^d \setminus \{0\}$. Show that there exist constants α and β with $0 \leq \alpha \leq \beta \leq 1$ such that for each $0 \leq g \in C_0(\mathbb{R}^d)$, we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^d} f_n(x) g(x)\, dx = \alpha g(0), \quad \limsup_{n\to\infty} \int_{\mathbb{R}^d} f_n(x) g(x)\, dx = \beta g(0).$$

Problem 5. For $f \in C[0,1]$ we define

$$L_n(f) := \sum_{k=1}^n a_{kn} f(x_{kn}), \quad n = 1, 2, \ldots,$$

where $0 \leq x_{1n} < x_{2n} < \ldots < x_{nn} \leq 1$ and $a_{kn} \in \mathbb{R}$. Show that

$$L_n(f) \to \int_0^1 f(x)\, dx \quad \text{as } n \to \infty$$

for each $f \in C[0,1]$ if and only if this holds for $f(x) = x^j$, $j = 0, 1, 2, \dots$, and

$$\sum_{k=1}^{n} |a_{kn}| \leq M, \quad n = 1, 2, \dots,$$

for some constant $M > 0$.

Problem 6. Let $E \subseteq \mathbb{R}$ be a measurable subset of $\mathbb{R}$ such that $E = E + 1/n$ for all $n = 1, 2, \dots$. Show that $m(E) = 0$ or $m(\mathbb{R} \backslash E) = 0$, where m is Lebesgue measure on $\mathbb{R}$.

Problem 7. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and suppose $f : \mathbb{D} \to \mathbb{C}$ is an injective holomorphic function with $f(0) = 0$ and $f'(0) = 1$. Define $\Omega := f(\mathbb{D})$.

a) Show that $\Omega \neq \mathbb{C}$ and $\partial\Omega \neq \emptyset$.
b) Show that $\text{dist}(0, \partial\Omega) := \inf\{|z| : z \in \partial\Omega\} \leq 1$.

Problem 8. By definition an analytic arc in the plane is the image $h(I)$ of an open interval $I \subset \mathbb{R}$ under a function h that is holomorphic and injective on an open set U with $I \subset U \subset \mathbb{C}$. Let Ω be a simply connected open set in the plane and let $\gamma \subset \partial\Omega$ be an analytic arc. Suppose $F : \mathbb{D} \to \Omega$ is a conformal mapping from $\mathbb{D}$ onto Ω and assume F has an extension to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. Show that there is an open set $W \supseteq \mathbb{D} \cup F^{-1}(\gamma)$ and a holomorphic function G on W such that $G = F$ on $\mathbb{D}$.

Problem 9. Let H be the Hilbert space of functions $f(z)$ holomorphic in $z = x + iy \in \mathbb{D}$ such that

$$||f||^2 := \int_{\mathbb{D}} |f(z)|^2 dxdy < \infty,$$

with inner product

$$\langle f, g \rangle := \int_{\mathbb{D}} f(z)\overline{g(z)} dxdy.$$

Let $\alpha \in \mathbb{D}$. Show that $H \ni f \mapsto f'(\alpha) \in \mathbb{C}$ is a bounded linear functional on H and exhibit $g_\alpha \in H$ such that $f'(\alpha) = \langle f, g_\alpha \rangle$, for all $f \in H$.

Problem 10. Let $\{f_n\}$ be a sequence of holomorphic functions on $\mathbb{D}$. Suppose that there exists $M \geq 0$ such that $|f_n(z)| \leq M$ for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$ and that the sequence $\{f_n(1/k)\}_{n \in \mathbb{N}}$ converges for each $k \in \mathbb{N}$, $k \geq 2$. Show that there exists a holomorphic function h on $\mathbb{D}$ such that $f_n \to h$ locally uniformly on $\mathbb{D}$ as $n \to \infty$.

Hint: For logical clarity and a readable presentation it helps to avoid multiple subscripts in the notation for subsequences and instead to write "let $\{g_n\}$ be subsequence of $\{f_n\}$" or something similar whenever this is appropriate.

Problem 11. Let $\Omega \subseteq \mathbb{C}$ be an open and connected set and let $h : \Omega \to \mathbb{R}$ be a harmonic function. Show that if h vanishes on a set of positive measure in Ω, then $h = 0$.

Problem 12. Let $U \subseteq \mathbb{C}$ be a bounded open set and $u : U \to \mathbb{R}$ be a continuous subharmonic function.

a) Show that if $M \in \mathbb{R}$ is a constant such that

$$\limsup_{z \to z_0} u(z) \le M \tag{1}$$

for all $z_0 \in \partial U$, then $u(z) \le M$ for all $z \in U$.

b) Show that if u is bounded from above and there exists a finite set $F \subseteq \partial U$ such that (1) is valid for all $z_0 \in \partial U \setminus F$, then the conclusion of (a) is still true, that is, we have $u(z) \le M$ for all $z \in U$.